On the Maximum Enstrophy Growth in Burgers Equation

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Abstract. The regularity of solutions of the three–dimensional Navier–Stokes equation is controlled by the boundedness of the enstrophy $\mathcal{E}$. The best estimate available to–date for its rate of growth is $d\mathcal{E}/dt \leq C\mathcal{E}^3$, where $C > 0$, which was recently found to be sharp by Lu & Doering (2008). Applying straightforward time–integration to this instantaneous estimate leads to the possibility of loss of regularity in finite time, the so–called "blow–up", and therefore the central question is to establish sharpness of such finite–time bounds. We consider an analogous problem for Burgers equation which is used as a "toy model". The problem of saturation of finite–time estimates for the enstrophy growth is stated as a PDE–constrained optimization problem

$$\max_{\phi}[\mathcal{E}(T) - \mathcal{E}(0)] \quad \text{subject to} \quad \mathcal{E}(0) = \mathcal{E}_0$$

where the control variable ϕ represents the initial condition, which is solved numerically for a wide range of time windows $T > 0$ and initial enstrophies $\mathcal{E}_0$. We find that the maximum enstrophy growth in finite time scales as $\mathcal{E}_0^\alpha$ with $\alpha \approx 3/2$. The exponent is smaller than $\alpha \approx 3$ predicted by analytic means, therefore suggesting lack of sharpness of analytical estimates.

1. Introduction

In this investigation we are interested in the largest enstrophy growth that can be achieved in a hydrodynamic system with some fixed initial enstrophy $\mathcal{E}_0$. This question is motivated by one of the "millennium problems" of the Clay Mathematics Institute (Fefferman, 2006), namely, whether the three–dimensional Navier–Stokes equation with *smooth* initial condition at $t = 0$ admits smooth solutions for all times $t > 0$. In other words, the question is whether a finite–time "blow–up" could occur in the Navier–Stokes system starting from some arbitrary smooth initial data. While it is well known that boundedness of the enstrophy $\mathcal{E}(t) = \int_\Omega \mathbf{u}(t)^2 \, d\Omega$ implies smoothness of the solution $\mathbf{u}(t)$, the best estimate for the rate of growth of enstrophy is $d\mathcal{E}(t)/dt < C\mathcal{E}(t)^3$ for some constant $C > 0$. There is recent computational evidence by Lu & Doering (2008) showing that this estimate is in fact sharp at any *single* instant of time. The central question is therefore how to extend this estimate to finite time intervals $(0, T]$, where $T > 0$, in a way that accounts for the constraint of the system evolution. We note that a straightforward time integration of the above bound for $d\mathcal{E}(t)/dt$ leads to a finite–time blow–up of the enstrophy, namely

$$E(t) \leq \frac{E(0)}{\sqrt{1 - 4\frac{CE(0)^2}{\nu^3}t}}. \tag{1}$$

Therefore, a long–term goal of our research is to use computational methods of PDE–constrained optimization to determine to what extent these finite–time estimates can be saturated by the actual evolution of the flow.

2. Model Problem

As a first step we have addressed this question in the context of the one–dimensional Burgers equation in a periodic domain for which the corresponding estimate of the enstrophy growth rate takes the form $d\mathcal{E}(t)/dt < c\,\mathcal{E}(t)^{\frac{5}{3}}$ for some constant $c > 0$. This instantaneous estimate is also known to be sharp and the corresponding maximizing solutions were found in Lu & Doering (2008). Integrating this estimate over $[0, T]$ and using some standard bounds one obtains (Doering, 2010)

$$\max_{t\in[0,T]} \mathcal{E}(t) \leq [\mathcal{E}_0^{\frac{1}{3}} + c'\mathcal{E}_0]^3, \tag{2}$$

where c' is a positive constant. Given smooth initial data, the viscous Burgers equation is known to lead to smooth solutions valid for all times, however, the question how well theoretical estimate (2) is saturated by actual solutions of the equation is still quite relevant, because bounds (1) and (2) are obtained using similar methods. We add that using the instantaneously–optimal solutions found in Lu & Doering (2008) one obtains the scaling $\max_{t\in[0,T]} \mathcal{E}(t) \sim \mathcal{E}_0^1$ in the limit of large initial enstrophies $\mathcal{E}_0$. Since the exponent is unity, this power–law is rather far from saturating theoretical estimate (2). In order to address this issue we have attempted to compute solutions of the Burgers system which, for a given initial enstrophy $\mathcal{E}_0$ and time window $[0, T]$, may saturate the bound (2). This is done by solving numerically a family of PDE–constrained optimization problems of the form

$$\max_{\phi\in H^1(\Omega)} \mathcal{E}(T)$$
$$\text{subject to} \quad \mathcal{E}(0) = \mathcal{E}_0, \tag{3}$$

where ϕ is the initial condition for Burgers equation, for a broad range of initial enstrophies $\mathcal{E}_0$ and time windows $(0, T]$ covering several orders of magnitude. The condition $\phi \in H^1(\Omega)$ implies that the initial data, which is our control variable, should belong to a suitable Sobolev space of functions with square–integrable derivatives. For all parameter values the initial conditions ϕ which correspond to the maxima in (3) are found using an iterative gradient–ascent method. A central element of this approach is determination of the cost functional gradient which is done using a suitably-defined adjoint system (Ayala, 2010). Another important element is "arc–minimization" used to determine the length of the maximization step in a way ensuring that the constraint $\mathcal{E}(0) = \mathcal{E}_0$, cf. (3), is satisfied up to the machine accuracy.

3. Results and Conclusions

Great care was exercised to make sure that all maxima of problem (3) are identified and computational evidence was found for the presence of an infinite, but countable, number of local maxima. All these local maximizers, however, turn out to be rescaled copies of only one solution. Such local maximizers corresponding to the two lowest wavenumbers are shown in Figure 1a. The rescalings, which leave Burgers equation invariant, are parametrized by the dominating wavenumber of the solution. Therefore, after such rescaling is applied, the different local maximizers exhibit in fact the same behavior of $\max_{t\in[0,T]} \mathcal{E}(t)$ vs. $\mathcal{E}_0$. It should be emphasized, however, that the presence of other local maximizers cannot be ruled

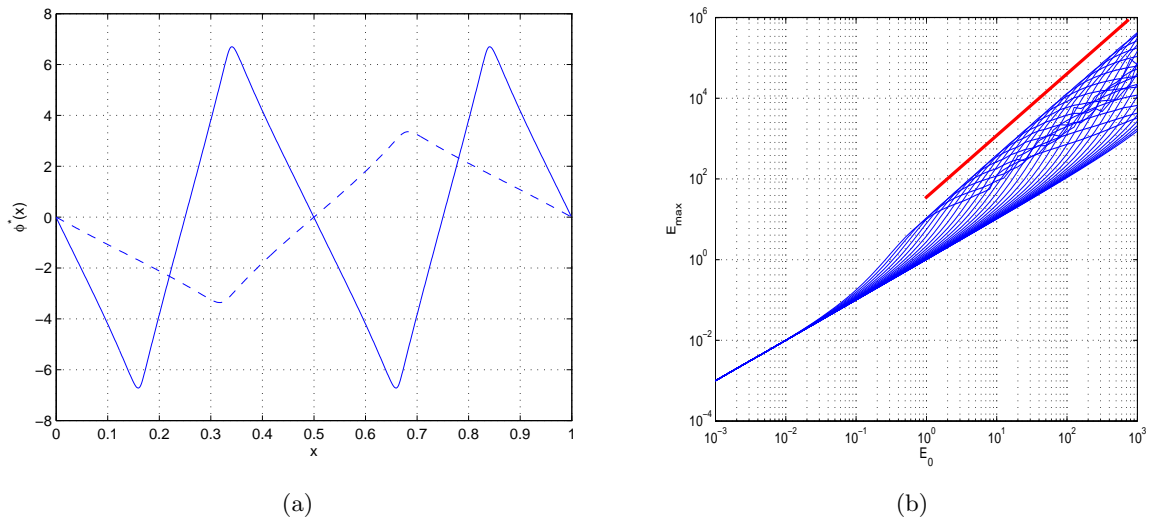(a)   (b)

Figure 1. (a) Two local maximizers of problem (3) which after rescaling exhibit the same scaling in $\max_{t\in[0,T]} \mathcal{E}(t) \sim \mathcal{E}_0^{\alpha}$, (b) dependence of $\max_{t\in[0,T]} \mathcal{E}(t)$ on $\mathcal{E}_0$ for different lengths T of the time window with a clearly visible envelope (marked in red).

	theoretical estimate (2)	optimal (instantaneous) [Lu & Doering, 2008]	optimal (finite–time) [present study]
α	3	1	3/2

Table 1. Exponents characterizing the power–law scaling of the maximum enstrophy built up in finite time $\max_{T>0}\left[\max_{\phi\in H^1(\Omega)} \mathcal{E}(T)\right]$ versus the initial enstrophy $\mathcal{E}_0$ in the limit of large $\mathcal{E}_0$.

out on purely computational grounds. The scaling of $\max_{t\in[0,T]} \mathcal{E}(t)$ versus $\mathcal{E}_0$ obtained for a range of different time horizons T is shown in Figure 1b. The main finding is that, as predicted by estimate (2), the maximum enstrophy indeed exhibits two distinct power–laws in the form $\max_{T>0}\left[\max_{\phi\in H^1(\Omega)} \mathcal{E}(T)\right] \sim \mathcal{E}_0^{\alpha}$ corresponding to the limits of small and large initial enstrophies $\mathcal{E}_0$. While for small $\mathcal{E}_0$ the exponent of the power–law is the same as predicted by (2), i.e., $\alpha = 1$, more importantly, the exponent corresponding to large $\mathcal{E}_0$ was found to be $\alpha \approx \frac{3}{2}$ which is significantly less than 3 predicted by theoretical relation (2). This suggests that the extension of the instantaneous estimate for the enstrophy growth rate $d\mathcal{E}(t)/dt < c\,\mathcal{E}(t)^{\frac{5}{3}}$ to finite times via a straightforward time integration is not sharp, and therefore the system evolution should be accounted for using a more accurate approach. As a summary, different exponents describing the power law $\max_{T>0}\left[\max_{\phi\in H^1(\Omega)} \mathcal{E}(T)\right] \sim \mathcal{E}_0^{\alpha}$ in the limit of large initial enstrophies $\mathcal{E}_0$ are collected in the Table 1.

Since for smooth initial data solutions of the viscous Burgers equations at any time $t \in (0,T]$ are real–analytic, we can also characterize their smoothness by examining the time–evolution of the associated width of the analyticity strip in the complex plane. This is done by tracking the location of the complex–plane singularities nearest to the real line (Sulem et al., 1982). This characterization opens up the possibility of a fundamentally different formulation of the problem of saturation of enstrophy estimates.

A complete description of the problem of the maximum enstrophy growth in a hydrodynamic system together with a detailed analysis of the computational optimization results obtained for Burgers equation is presented in Ayala & Protas (2011). Our future work will involve an analysis of the sharpness of the corresponding estimates available for incompressible flows in two dimensions and eventually in three dimensions. We remark that, as regards two–dimensional flows, the relevant quantity is the palinstrophy (i.e., the L_2 norm of the vorticity gradient), rather than enstrophy.

Acknowledgments

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