Advances in Turbulence X
Proceedings of the Tenth European Turbulence Conference
H. I. Andersson & P.-Å. Krogstad (Eds.)


# Reduced–Order Optimal Feedback Control of Transition in Bluff Body Wake Flows

B. Protas[1]
[1]Department of Mathematics & Statistics
McMaster University, Hamilton, ON, Canada
Contact address: *bprotas@mcmaster.ca*

## 1 The Föppl Model and Wake Instabilities

The last decade has witnessed a surge of attempts aiming at integration of Control Theory and Computational Fluid Dynamics (see Bewley [1] for a review). Despite encouraging success of many developed approaches, these studies have revealed extreme computational complexity of algorithms designed for systems governed by the full Navier–Stokes equation. These observations motivate the quest for reduced order models which can be used as a basis for development of more feasible control and optimization strategies. The objective of the present investigation is to provide a control–theoretic characterization of a very simple point vortex model for the purpose of control of instabilities in bluff body wakes.

The Föppl model [2] consists of two point vortices with opposite circulations which are added symmetrically, together with their images, above and below the centerline of the potential flow solution (Fig. 1a). Denoting $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (where $i = \sqrt{-1}$) the positions of the two vortices, $\Gamma_1 = -\Gamma$ and $\Gamma_2 = \Gamma$ (where $\Gamma > 0$) their circulations, the total velocity induced by this system (including images) is given by

$$V(z) = 1 + \frac{1}{z^2} - \frac{\Gamma_1}{2\pi i}\left(\frac{1}{z-z_1} - \frac{1}{z-1/\overline{z}_1}\right) + \frac{\Gamma_2}{2\pi i}\left(\frac{1}{z-z_2} - \frac{1}{z-1/\overline{z}_2}\right). \qquad (1)$$

The stationary points of this nonlinear dynamic system, i.e., the positions $z_0$ where $V(z_0) = \frac{dz}{dt}|_{z=z_0} = 0$, can be found analytically for every value of the circulation $\Gamma$. Such a steady Föppl system is characterized by the presence of a recirculation bubble reminiscent of the bubble appearing in the base flows of actual unstable wakes (Fig. 1b). The Föppl system also has interesting properties as regards stability of the stationary solution ([3]). These properties can be studied in the usual way, i.e., by perturbing the stationary solution and then analyzing the linearized evolution of the perturbations governed by the system (cf. Fig. 1c)

$$\frac{d\mathbf{X}'}{dt} = \mathcal{A}\mathbf{X}', \text{ where } \mathbf{X}' = [x_1', y_1', x_2', y_2']^T \text{ and } \mathcal{A} = \frac{\partial \mathbf{V}}{\partial_X}\Big|_{z=z_0}. \qquad (2)$$

Eigenvalue analysis of the operator $\mathcal{A}$ reveals the presence of the following modes (Fig. 1d): an unstable mode (corresponding to a real positive eigenvalue $\lambda_1 = \lambda_r$), a stable mode (corresponding to a real negative eigenvalue $\lambda_2 = -\lambda_r$), and a neutrally

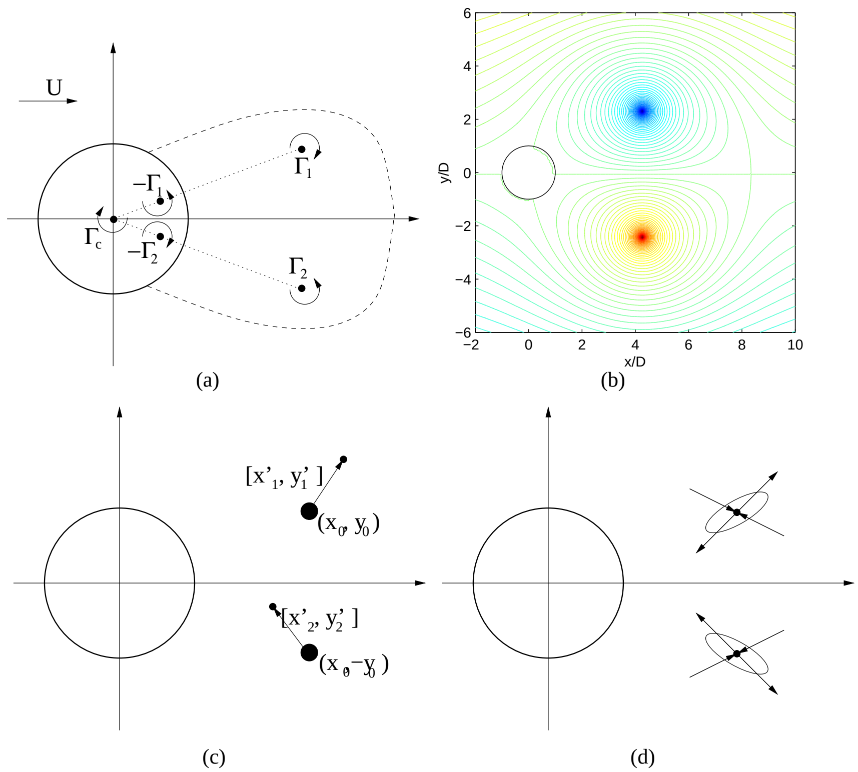


Figure 1: Characterization of the Föppl system: (a) positions of the point vortices and their images, (b) streamline pattern of the stationary solution, (c) perturbations to the stationary solution, and (d) various eigenmodes of the linearized system (2).

stable oscillatory mode (corresponding to a conjugate pair of purely imaginary eigenvalues $\lambda_{3/4} = \pm i\lambda_i$). By analyzing the spatial structure of the unstable eigenmodes Tang & Aubry [3] showed that the instability of the Föppl closely resembles the onset of vortex shedding in an actual cylinder wake undergoing Hopf bifurcation. In the case of wake flows the objective of most control strategies is to mitigate, if not eliminate completely, vortex shedding and thereby stabilize the unstable symmetric flow. Therefore, we propose here to use the Föppl system as a *reduced–order model* of the steady wake, with the onset of the vortex shedding instability modeled by the perturbed Föppl system. This reduced–order model will allow us to design a simple Linear–Quadratic–Gaussian (LQG) compensator which can be used for feedback control of the actual wake flow.

# 2 LQG Control Design

We outline here the design of a Linear–Quadratic–Gaussian compensator that uses the linearized Föppl system as a reduced–order model for the vortex shedding instability. The flow actuation $u$ has the form of cylinder rotation and is represented by adding a single control vortex $\Gamma_c$ at the origin (Fig. 1a). The control objective is to suppress vortex shedding by reducing the concomitant flow asymmetry which can be characterized using velocity measurements on the centerline $\mathbf{y} = \mathbf{V}|_{(x_m,0)}$ as the system output. Our reduced–order model is assumed uncertain and the modeling errors are represented by the plant noise $w$. The system output $\mathbf{y}$ is also assumed to be contaminated with noise $\mathbf{v}$. The feedback control $u$ is determined based on the instantaneous state of the system $\mathbf{X}'$, or its estimate $\mathbf{X}'_e$, none of which are a priori available. However, at a given time, an estimate $\mathbf{X}'_e$ can be deduced based on the measurement history. Consequently, solution of both estimation and control problems is required. The *estimator* uses the actual plant measurements $\mathbf{y}$ to reconstruct the corresponding state of the model (i.e., the state of the linearized Föppl system), which is in turn used by the *controller* to determine an instantaneous control $u$. An estimator and a controller combined together are referred to as *compensator*. Under certain standard assumptions regarding the stochastic parameters, the control and estimation problems can be solved independently, a property known as *the separation principle*. Formally, the estimation and control problems can be stated as follows ([4]):

- Find a stabilizing feedback control $u = \mathcal{K}\mathbf{X}'$ which minimizes the cost functional

$$\mathcal{J}(u) = \frac{1}{2}\int_0^\infty (\mathbf{y}^* \mathcal{Q}\mathbf{y} + uRu)\,dt, \quad \text{where} \quad \begin{cases} \dfrac{d\mathbf{X}'}{dt} = \mathcal{A}\mathbf{X}' + \mathcal{B}u + \mathcal{G}w, \\ \mathbf{y} = \mathcal{C}\mathbf{X}' + \mathcal{D}u + \mathcal{H}w + \mathbf{v}, \end{cases} \tag{3}$$

  with $R > 0$ and where $\mathcal{Q} \geq 0$, $\mathcal{B}$, $\mathcal{G}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{H}$ are matrices.

- Based on available measurements $\mathbf{y}(t)$ find an estimate $\mathbf{X}'_e$ of the state of the system which will minimize the weighted variance of the estimation error residual $[\mathbf{X}'_e - \mathbf{X}']^T \mathcal{G}\mathcal{W}\mathcal{G}^T [\mathbf{X}'_e - \mathbf{X}']$, where $E[w(t)w(\tau)^T] = \mathcal{W}\delta(t-\tau)$.

Both problem involve solution of an algebraic Riccati equation which in the case of the present simple problem can be obtained using standard tools.

# 3 Computational Results and Conclusions

Here we present computational results for two cases utilizing the LQG compensator introduced above: the control of an instability of the Föppl system in the fully nonlinear regime [Figs. 3(a-b)] and the control of the vortex shedding instability in the simulated 2D cylinder wake flow at $Re = 75$ [Figs. 3(c-e)]. In both cases the downstream position of the vortices in the stationary configuration was chosen at $x_0 = 4.32$ which results in the same recirculation length as in the actual base flow at $Re = 75$. The downstream position of the velocity measurement point is $x_m = 5.53$ which maximizes observability of the Föppl system. Results presented in Fig. 3 show that the LQG compensator successfully stabilizes both the Föppl system and the actual wake flow. We want to emphasize the remarkable simplicity of the control design based on a reduced–order model with four degrees of freedom only. Further details of this study will be reported in a forthcoming paper.

*The author wishes to thank PMMH / ESPCI in Paris for hospitality during the course of this work. Funding from CNRS is gratefully acknowledged.*

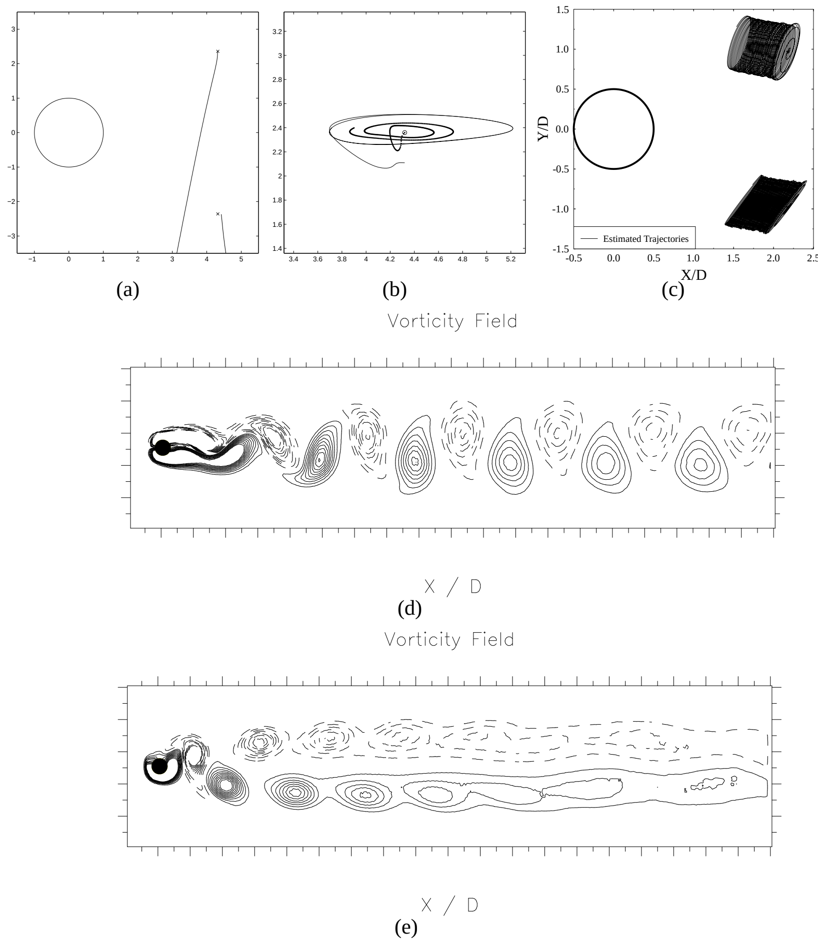


Figure 2: (a) Nonlinear instability of the Föppl system, (b) LQG control of the Föppl system (magnification, thin line — actual positions of the vortices, thick line — corresponding estimated positions); (c) estimated positions of the Föppl vortices in the control of an actual wake flow; (d) vorticity field in an uncontrolled wake flow; (e) vorticity field in the wake flow with an LQG feedback control.

# References

[1] Bewley, T.R., 2001, Flow control: new challenges for a new Renaissance, *P*rogress in Aerospace Sciences **37**, 21-58.

[2] Föppl, L., 1913, Wirbelbewegung hinter einem Kreiscylinder, *S*itzb. d. k. Bayr. Akad. d. Wiss. 1-17.

[3] Tang, S., Aubry, N., 1997, On the symmetry breaking instability leading to vortex shedding, *P*hysics of Fluids **9**, 2550-2561.

[4] Stengel, R.F., 1994, Optimal Control and Estimation, Dover Publications.