On Calculation of Hydrodynamic Forces for Steady Flows in Unbounded Domains

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Abstract

This note addresses the question why the "impulse formula", often employed to compute hydrodynamic forces in vortex–dominated time–dependent flows, is not applicable to steady flows in unbounded domains. By analyzing the asymptotic structure of steady and unsteady flow solutions in unbounded domains, it is demonstrated that one assumption made in the derivation of the impulse formula is in fact not satisfied in the steady case. This result also highlights the special character of steady flows in unbounded domains.

Keywords: hydrodynamic forces, impulse formula, steady flows, unbounded domains

1. Introduction

In this note we are concerned with a family of approaches to the calculation of hydrodynamic forces in flows past obstacles based on the so–called "impulse formula". We focus on flows in *unbounded* domains and seek to identify the reasons why, somewhat paradoxically, such formulations are not applicable to steady–state problems. We thus begin by considering solutions of the Navier–Stokes system describing the motion of viscous incompressible

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fluid

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} - \mu\boldsymbol{\Delta}\mathbf{u} + \boldsymbol{\nabla}p = 0 && \text{in } \Omega\times(0,T], && \text{(1a)}\\
&\boldsymbol{\nabla}\cdot\mathbf{u} = 0 && \text{in } \Omega\times(0,T], && \text{(1b)}\\
&\mathbf{u}\big|_{t=0} = \mathbf{u}_0 && \text{in } \Omega, && \text{(1c)}\\
&\mathbf{u}\big|_{\partial B} = 0 && \text{in } (0,T], && \text{(1d)}\\
&\mathbf{u} \longrightarrow U_\infty \mathbf{e}_1 && \text{in } (0,T] \text{ for } |\mathbf{x}|\to\infty, && \text{(1e)}
\end{aligned}
$$

where $T < \infty$, B is a finite solid body and ∂B its boundary, $\Omega = \mathbb{R}^N\backslash\overline{B}$ is the (unbounded) flow domain ($N = 2, 3$ is the space dimension), μ is the viscosity of the fluid, whereas $\mathbf{u} = [u_1, \dots, u_N]^{\mathrm{T}}$ and p represent the fluid velocity and pressure. Without loss of generality, in system (1) it is assumed that the fluid density is equal to the unity. The coordinate system is attached to the obstacle B, and to fix attention we assume that the flow at infinity is constant and aligned with the OX axis of the coordinate system, cf. (1e), where $U_\infty \in \mathbb{R}^+$ and $\mathbf{e}_1$ is the corresponding unit vector. The initial condition is given by $\mathbf{u}_0$ and for simplicity we assume that the no–slip and no–penetration boundary conditions apply on the body B, cf. (1d).

Efficient calculation of hydrodynamic forces acting on the body is a challenging problem and application of the definition formula

$$
\mathbf{F} \triangleq \oint_{\partial B} (-p\,\mathbf{n} + \mathbf{n}\cdot\boldsymbol{\Pi})\, d\sigma, \tag{2}
$$

where $\boldsymbol{\Pi} \triangleq \mu\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right]$ is the viscous stress tensor, $\mathbf{n}$ is the unit vector normal to the obstacle boundary directed out of the flow domain (Fig. 1), and "$\triangleq$" means "equal to by definition", is usually not an optimal solution from the practical point of view both in numerical computations and in analysis of experimental data. There are several alternative approaches, see, e.g., Wu et al. (2006) for a survey, and one technique which has received some attention in the literature is based on the "impulse formula" (Batchelor, 1967; Biesheuvel and Hagmeijer, 2006; Saffman, 1992),

$$
\mathbf{F} = -\frac{1}{N-1}\frac{d}{dt}\int_\Omega \mathbf{x}\times\boldsymbol{\omega}\, d\Omega, \tag{3}
$$

where $\boldsymbol{\omega} \triangleq \boldsymbol{\nabla}\times\mathbf{u}$ is the vorticity and $\mathbf{x} = [x_1, \dots, x_N]^{\mathrm{T}}$ is the position vector. In addition to the absence of pressure, an advantage of formula (3) is that it relies on a highly localized quantity such as vorticity and therefore provides an explicit connection between the vortex dynamics in the flow and the hydrodynamic forces. Owing to these properties, formula (3) and its different variants (Noca et al., 1999) are attractive approaches to calculation of hydrodynamic forces in flows in which strong vorticity occupies only a small portion of the flow domain. Such approaches are therefore particularly useful in situations where high–resolution vorticity fields are available, but it is not straightforward to measure or compute the corresponding pressure fields. Thus, formula (3) and its variants have been often used in flow calculations based on vortex methods (e.g., Koumoutsakos and Leonard, 1995; Noca et al., 1997; Protas et al., 2000; Shiels et al., 1996) and in laboratory experiments involving Digital Particles Image Velocimetry (e.g., Birch et al., 2004; Dabiri, 2005; Thiria et al., 2006, and references quoted therein).

In view of the advantages discussed above, it might be tempting to apply formula (3) to *steady* problems, i.e., flows satisfying system (1) in which the time–derivative term $\partial \mathbf{u}/\partial t$ is dropped in equation (1a) and equation (1c) is eliminated. However, we observe that it would, paradoxically, give a wrong result, namely, that the hydrodynamic force (including drag) is identically zero which is obviously not the case (Fornberg, 1985). This result is independent of the Reynolds number which is typically defined as $\mathrm{Re} \triangleq \mathrm{diam}(B)\, U_\infty/\mu$, where $\mathrm{diam}(B)$ is the characteristic dimension of the obstacle in the direction perpendicular to the flow (we recognize that such steady–state flows tend to be unstable for large Reynolds numbers, however this issue is irrelevant for the present discussion). Since, to the best of the author's knowledge, this issue has never been addressed in the literature, the goal of this note is to identify the reasons for this paradox. We will recall results from the mathematical literature which demonstrate that such steady–state solutions of system (1) have in fact different asymptotic properties at infinity than the time–dependent solutions. As a result, certain essential assumptions made in the derivation of formula (3) are not satisfied by the steady–state solutions rendering this formula inapplicable in such cases. In the next Section we review the derivation steps leading to formula (3) emphasizing the assumptions made along the way as regards the behavior of solutions of system (1) at infinity. In that Section we will refer to results available in the mathematical literature to indicate how the steady and unsteady solutions differ in this regard. Finally, in Section 3 we will discuss the significance of this result and draw some conclusions.

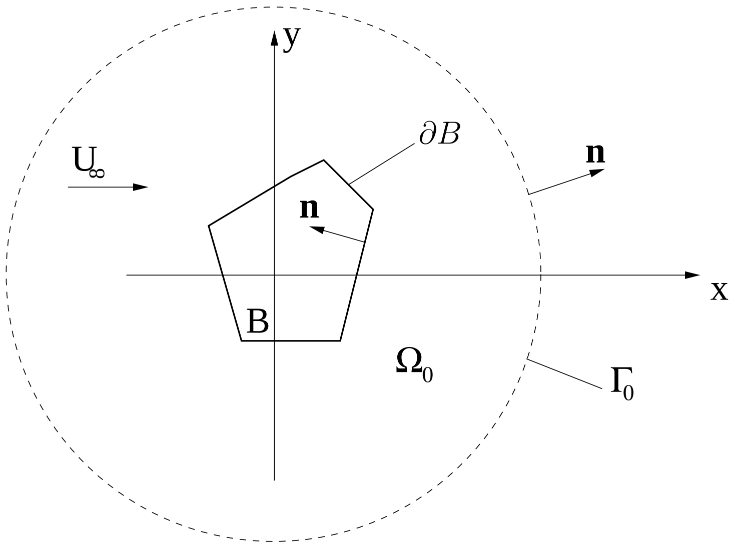

Figure 1: Schematic of the flow past an obstacle B in an unbounded exterior domain which also indicates the finite control volume Ω_0 with the boundary Γ_0.

2. Derivation of Impulse Formula

Impulse formula (3) is derived in four main steps which we summarize below; the reader is referred to the monograph by Wu et al. (2006) and the papers by Noca et al. (1999) and by Graziani and Bassanini (2002) for additional technical details

1. first, we consider momentum balance in a *finite* domain Ω_0 exterior to the body B and bounded by the surface (in 3–D) or contour (in 2–D) Γ_0, see Fig. 1,
$$\mathbf{F} = -\frac{d}{dt}\int_{\Omega_0} \mathbf{u}\, d\Omega - \oint_{\Gamma_0} (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\, d\sigma + \oint_{\Gamma_0} (-p\mathbf{n} + \mathbf{n}\cdot\boldsymbol{\Pi})\; d\sigma, \quad (4)$$
2. next, we transform the momentum integral in expression (4) using the "derivative–moment transformation" (Wu et al., 2006) which expresses an integral of a vector field in terms of an integral of a moment of a corresponding derivative field and a suitable boundary term, namely
$$\int_{\Omega_0} \mathbf{u}\, d\Omega = \frac{1}{N-1}\int_{\Omega_0} \mathbf{x}\times\boldsymbol{\omega}\, d\Omega - \frac{1}{N-1}\oint_{\partial\Omega_0} \mathbf{x}\times(\mathbf{n}\times\mathbf{u})\, d\sigma, \quad (5)$$

where $\partial\Omega_0 \triangleq \partial B \bigcup \Gamma_0$,

3. then, combining expressions (4) and (5), and using some vector identities (Graziani and Bassanini, 2002; Noca et al., 1999) to eliminate pressure we can recast the force as

$$\mathbf{F} = -\frac{1}{N-1}\frac{d}{dt}\int_{\Omega_0} \mathbf{x} \times \boldsymbol{\omega}\, d\Omega + \oint_{\Gamma_0} \mathbf{n} \cdot \boldsymbol{\gamma}\; d\sigma \tag{6}$$

in which the tensor field $\boldsymbol{\gamma}$ has the form (Noca et al., 1999; Graziani and Bassanini, 2002)

$$\begin{aligned}\boldsymbol{\gamma} =& \mu\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right] - \mathbf{u}\mathbf{u} + \frac{1}{2}|\mathbf{u}|^2\mathbf{I} \\ &+ \frac{1}{N-1}\left[(\mathbf{x}\times\mathbf{u})\boldsymbol{\omega} - \mathbf{u}(\mathbf{x}\times\boldsymbol{\omega}) - \mu(\mathbf{x}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\omega})\mathbf{I} - \mathbf{x}(\boldsymbol{\nabla}\times\boldsymbol{\omega}))\right],\end{aligned} \tag{7}$$

where $\mathbf{I}$ is the identity matrix,

4. finally, we take the limit $\Gamma_0 \to \infty$ in expression (6) such that Ω_0 encloses all of Ω in this limit; *assuming* a sufficiently rapid decay of the velocity and vorticity fields $\mathbf{u}$ and $\boldsymbol{\omega}$ towards their asymptotic values, relation (6) then yields formula (3).

This assumed asymptotic behavior of the velocity and vorticity fields at infinity is our main concern here, and below we examine the validity of these assumptions in the steady and unsteady case, respectively. To fix attention, we will hereafter focus on the two–dimensional (2–D) case, however, the analysis and conclusions are quite similar in three dimensions (3–D). Thus, unless stated otherwise, estimates will be provided for the 2–D case.

As regards time–dependent flows, the following asymptotic behavior is rigorously established by Mizumachi (1984, Theorem 1)

$$|\mathbf{u} - U_\infty \mathbf{e}_1| \sim \mathcal{O}\left(\frac{1}{r^N}\right) \qquad \text{as } r \to \infty, \tag{8}$$

where $r \triangleq |\mathbf{x}|$. It has been, in addition, assumed that (Graziani and Bassanini, 2002)

$$\left|\frac{\partial u_i}{\partial x_j}\right| \sim \mathcal{O}\left(\frac{1}{r^N}\right),\ i,j = 1,\ldots,N \qquad \text{as } r \to \infty. \tag{9}$$

As concerns vorticity, we have for finite times $T < \infty$ (Wu, 1980)

$$|\boldsymbol{\nabla} \times \mathbf{u}| \sim \mathcal{O}\left(\mathrm{e}^{-mr}\right) \tag{10}$$

for some $m > 0$. Properties (8), (9) and (10) are explicitly assumed, or implied, in typical derivations of impulse formula (3), see, e.g., Biesheuvel and Hagmeijer (2006); Graziani and Bassanini (2002); Noca et al. (1999). Indeed, assuming that the initial condition $\mathbf{u}_0$ satisfies conditions (8)–(10), property (10) guarantees that the impulse integral $\int_{\Omega_0} \mathbf{x} \times \boldsymbol{\omega}\, d\Omega$ remains bounded in the limit $\Gamma_0 \to \infty$ for all $t < \infty$. Using additionally relations (8) and (9) we obtain (in 2–D) $|[\boldsymbol{\gamma}]_{i,j}| \sim 1 + \mathcal{O}(r^{-2})$, $i, j = 1, 2$, for $r \to \infty$, so that the integral $\oint_{\Gamma_0} \mathbf{n} \cdot \boldsymbol{\gamma}\, d\sigma$ indeed vanishes in the limit $\Gamma_0 \to \infty$, as stipulated in Step 4 of the derivation of the impulse formula presented above.

As regards time–independent flows, the asymptotic behavior of such solutions at infinity has received some attention in the mathematical literature since the work of Finn (1965) and Smith (1965), although in the author's opinion these studies have gone rather unnoticed in the fluid mechanics community. We refer the reader to the monograph by Galdi (1994) for a survey of these results. From the point of view of the present study, the most important result is that steady–state solutions of system (1) exhibiting features observed in reality such as nonzero drag, and hence referred to as "physically reasonable", have the following asymptotic behavior for $|\mathbf{x}| \to \infty$ in the 2–D case (Galdi, 1994, Theorem 6.1 in Volume II)

$$\mathbf{u}(\mathbf{x}) = U_\infty \mathbf{e}_1 + \mathbf{E}(\mathbf{x}) \cdot \mathbf{F} + \mathbf{W}(\mathbf{x}), \tag{11}$$

where $\mathbf{E}(\mathbf{x})$ is the Oseen fundamental tensor, whereas the field $\mathbf{W}(\mathbf{x})$ satisfies the estimate $|\mathbf{W}(\mathbf{x})| \sim \mathcal{O}(r^{-1+\epsilon_1})$ for some arbitrarily small $\epsilon_1 > 0$. The Oseen tensor $\mathbf{E}(\mathbf{x})$ is a fundamental solution of the Oseen equation obtained from the steady–state version of equations (1a)–(1b) by replacing the nonlinear advection term in equation (1a) with $U_\infty\,(\partial \mathbf{u}/\partial x_1)$. Construction of the Oseen tensor and its properties, especially in regard to the slow asymptotic decay at infinity, are reviewed in detail in the monograph by Galdi (1994, Volume I), and below we refer to the main results only. In view of the rapid decay of the field $\mathbf{W}(\mathbf{x})$, the behavior of the steady–state Navier–Stokes solutions in 2–D at large distances $\mathbf{x}$ is to the leading order the same as in the Oseen flows with the same reaction force $\mathbf{F}$ and this property holds regardless of the Reynolds number Re (in fact, the same is also true in the

3–D case, cf. Galdi (1994)). More specifically, such steady solutions feature a paraboloidal "wake" region with boundary described by the polar equation

$$r(1-\cos\varphi)=L, \tag{12}$$

where φ is the polar angle and $L>0$ a constant parameter (Fig. 2), in which the longitudinal velocity component u_1 exhibits a slow decay towards its asymptotic value, namely

$$|u_1(\mathbf{x})-U_\infty|\sim\mathcal{O}(r^{-1/2}) \quad \text{as } r\to\infty. \tag{13}$$

We add that, as demonstrated by Galdi (1994, Theorem 6.2 in Volume II), estimate (13) is in fact sharp, in the sense that there exists a constant $C>0$ such that

$$|u_1(\mathbf{x})-U_\infty|\geq\frac{C}{r^{\frac{1}{2}}} \quad \text{as } r\to\infty. \tag{14}$$

Outside the paraboloidal wake region the component u_1 has a faster decay, so that

$$|u_1(\mathbf{x})-U_\infty|\sim\mathcal{O}(r^{-(1/2+\epsilon_2)}) \quad \text{as } r\to\infty, \tag{15}$$

for some $\epsilon_2>0$ (Galdi, 1994). In 2–D (but not in 3–D) the transverse velocity component u_2 does not exhibit wake behavior and obeys the uniform bound $u_2\sim\mathcal{O}(r^{-(1+\epsilon_3)})$, $\epsilon_3>0$. There also exists an estimate for the decay of the vorticity field $\omega\triangleq\boldsymbol{\omega}\cdot\mathbf{e}_3$ away from the obstacle in steady 2–D Navier–Stokes flows, namely (Galdi, 1994, Theorem 6.4 in Volume II)

$$\omega(\mathbf{x})=\frac{\partial\Psi}{\partial x_1}(\mathbf{F}\cdot\mathbf{e}_2)-\frac{\partial\Psi}{\partial x_2}(\mathbf{F}\cdot\mathbf{e}_1)+V(\mathbf{x}), \tag{16}$$

where $\Psi(\mathbf{x})\triangleq e^{\frac{x_1\,\mathrm{Re}}{2}}\,\mathrm{K}_0(\frac{|\mathbf{x}|\,\mathrm{Re}}{2})$, with K_0 denoting the modified Bessel function of the second type of order zero, and $V(\mathbf{x})$ is a field satisfying the estimate $|V(\mathbf{x})|\sim\mathcal{O}(\mathrm{e}^{-\rho}\,|\mathbf{x}|^{-3/2}\log|\mathbf{x}|)$, where $\rho\triangleq(|\mathbf{x}|-x_1)$. We wish to emphasize at this point that relations (11) and (16), together with the resulting estimates (13) and (15), are not hypotheses or assumptions, but rigorously established mathematical facts about steady–state solutions of the Navier–Stokes system in externally unbounded domains in 2–D. We thus conclude that a distinguishing feature of such steady–state flows is the presence of a

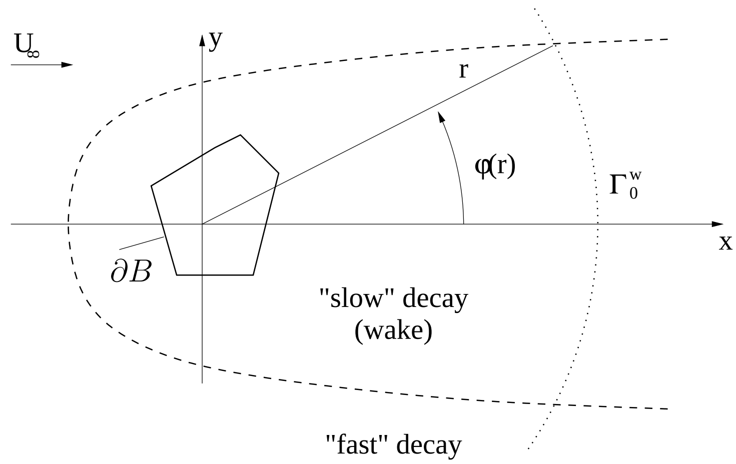

Figure 2: Schematic partition of the flow domain into the regions with "slow" decay (wake) and "fast" decay of the steady–state solutions of (1). Dashed line represents the schematic boundary between these two regions described by equation (12). Dotted line represents the part of the contour Γ_0 contained in the wake.

"wake" region extending downstream to infinity where the longitudinal velocity component decays less rapidly than outside this region. The decay of the steady–state solutions in the wake region is also significantly slower than in the corresponding time–dependent flows, cf. relations (8)–(10). As we shall see below, this fact has far–reaching consequences for the derivation of impulse formula (3).

We are now ready to revisit Step 4 in the derivation of impulse formula (3), this time concentrating on the steady flows. First, we observe that, in view of property (16), it is not possible to ascertain boundedness of the impulse integral $\int_{\Omega_0} \mathbf{x} \times \boldsymbol{\omega}\, d\Omega$ in the limit $\Gamma_0 \to \infty$. Secondly, we will consider the behavior of the contour integral $\oint_{\Gamma_0} \mathbf{n} \cdot \boldsymbol{\gamma}\, d\sigma$ and, to fix attention, we will focus on just one term, namely $\frac{1}{2} \oint_{\Gamma_0} |\mathbf{u}|^2 \mathbf{n}\, d\sigma$, cf. expressions (6)–(7). Following the same approach as below it can be shown that the other terms involving $\mathbf{u}$ and $\boldsymbol{\nabla}\mathbf{u}$ in the contour integral exhibit in fact analogous behavior. We split the contour Γ_0 into the part Γ_0^w contained in the wake (Fig. 2) and its complement $\Gamma_0 \backslash \Gamma_0^w$ which allows us to rewrite the contour integral as follows (we consider the X component of this vector–valued integral only, the

Y component being identically zero due to the symmetry of the flow with respect to the OX axis)

$$\frac{1}{2}\oint_{\Gamma_0} |\mathbf{u}|^2 \mathbf{n}\, d\sigma = \oint_{\Gamma_0^w} \left(|\mathbf{u}|^2 - U_\infty^2\right) \mathbf{n}\, d\sigma + \oint_{\Gamma_0 \setminus \Gamma_0^w} \left(|\mathbf{u}|^2 - U_\infty^2\right) \mathbf{n}\, d\sigma, \quad (17)$$

where we also used the fact that $U_\infty^2 \oint_{\Gamma_0} \mathbf{n}\, d\sigma = 0$. As regards the integral over Γ_0^w, using estimate (13) we thus obtain for sufficiently large Γ_0

$$\begin{aligned} \left| \int_{\Gamma_0^w} \left(|\mathbf{u}|^2 - U_\infty^2\right) \mathbf{n} \cdot \mathbf{e}_1\, d\sigma \right| &\geq \left| \int_{\Gamma_0^w} \frac{C}{r^{1/2}} \cos\varphi'\, d\sigma \right| \\ &= C r^{1/2} \left| \int_{-\varphi(r)}^{\varphi(r)} \cos\varphi'\, d\varphi' \right| = 2 C r^{1/2} \big| \sin\varphi(r) \big|, \end{aligned} \quad (18)$$

where the constant $C > 0$ is defined in estimate (14) and $\varphi(r)$ is the polar angle characterizing the boundary of the wake region for a given r, cf. Fig. 2. Using relation (12) and expanding in a series in terms of r we obtain

$$\sin\varphi(r) = \sin\left[\arccos\left(1 - \frac{L}{r}\right)\right] = \frac{D_1}{r^{1/2}} + \frac{D_2}{r^{3/2}} + \frac{D_3}{r^{5/2}} + \dots, \quad (19)$$

where D_i, $i = 1, 2, \dots$, are nonvanishing constants such that $D_1 > 0$ and whose actual numerical values are not important. We add that expansion (19) is centered at infinity, since it is obtained for $(1/r) \to 0$. Combining relations (18) and (19) we obtain

$$\left| \int_{\Gamma_0^w} \left(|\mathbf{u}|^2 - U_\infty^2\right) \mathbf{n} \cdot \mathbf{e}_1\, d\sigma \right| \geq \left| 2CD_1 + \frac{2CD_2}{r} + \frac{2CD_3}{r^2} + \dots \right| \underset{r\to\infty}{\longrightarrow} 2CD_1 \neq 0 \quad (20)$$

which means that the contribution to the integral $\frac{1}{2}\oint_{\Gamma_0} |\mathbf{u}|^2 \mathbf{n}\, d\sigma$ from the wake region Γ_0^w *does not* vanish as $\Gamma_0 \to \infty$. Computing in a similar way the contribution to this integral due to the remaining part of the contour $\Gamma_0 \setminus \Gamma_0^w$, which now involves a faster decay with estimate (15), we obtain for sufficiently large Γ_0

$$\left| \int_{\Gamma_0 \backslash \Gamma_0^w} \left(|\mathbf{u}|^2 - U_\infty^2 \right) \mathbf{n} \cdot \mathbf{e}_1 \, d\sigma \right| \le \left| \int_{\Gamma_0 \backslash \Gamma_0^w} \frac{C'}{r^{1/2+\epsilon}} \cos\varphi' \, d\sigma \right|$$
$$= C' \frac{r}{r^{1/2+\epsilon}} \left| \int_{\varphi(r)}^{2\pi-\varphi(r)} \cos\varphi' \, d\varphi' \right| = \frac{2C'}{r^{\epsilon}} \left| D_1 + \frac{D_2}{r} + \frac{D_3}{r^2} + \dots \right| \underset{r\to\infty}{\longrightarrow} 0, \tag{21}$$

where the constant $C' > 0$ is chosen so that $|u_1(\mathbf{x}) - U_\infty| \le C'/r^{1/2+\epsilon_2}$ outside the wake region for large r. Relation (21) implies that the contribution to the integral from the region outside the wake does vanish as $\Gamma_0 \to \infty$. Therefore, the contributions to the integral from Γ_0^w and $\Gamma_0 \backslash \Gamma_0^w$ cannot cancel each other in the limit $r \to \infty$ which allows us to conclude that in steady–state 2–D flows $\oint_{\Gamma_0} \mathbf{n} \cdot \boldsymbol{\gamma} \, d\sigma \to \text{Const} \neq 0$ as $\Gamma_0 \to \infty$, in contrast to what must be assumed in Step 4 of the derivation of impulse formula (3). As a result, this formula is not valid in steady flows in unbounded domains.

3. Conclusions

In this note we addressed the apparent paradox of incorrect results produced by impulse formula (3) when applied to steady flows. Using the results of rigorous mathematical analysis of steady–state solutions of the Navier–Stokes equation in unbounded domains, it was shown that in fact one step in the derivation of impulse formula (3) is not justified rendering this formula invalid in such problems. More specifically, the reason is the slow decay of the steady–state velocity and vorticity fields as compared to the time–dependent case, resulting in the flux integral $\oint_{\Gamma_0} \mathbf{n} \cdot \boldsymbol{\gamma} \, d\sigma$ not vanishing in the limit $\Gamma_0 \to \infty$. This result may appear counter–intuitive, since it is usually tempting to regard time–independent phenomena as "special cases" of time–dependent phenomena, in the sense that relations valid in the latter case should be also valid in the former case (possibly after setting time–derivatives to zero). The problem discussed in this note shows clearly that this is not necessarily the case. We also mention that the presence of the time derivative in formula (3) might from the beginning raise some doubts about application of this formula to the steady case. This issue is, however, more subtle, as in the derivation process (Section 2) one could absorb the time derivative into the impulse integral before taking the limit $\Gamma_0 \to \infty$, resulting in a formula containing the term $\int_{\Omega_0} \mathbf{u} \times \boldsymbol{\omega} \, d\Omega$ (i.e., without any explicit time differentiation). While such variations of the impulse formula have been considered in time–dependent flows (Birch et al., 2004; Noca et al., 1999), they remain inapplicable in the steady setting for the same reasons as described above. Finally, we remark that hydrodynamic forces in steady flows in unbounded domains can be conveniently determined by considering the momentum balance in a *finite* control volume, as discussed for example in Wu et al. (2006).

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