From the Velocity and Vorticity Fields to Hydrodynamic Forces — A Survey of Mathematical and Computational Approaches

Bartosz Protas[1,a]

Department of Mathematics & Statistics, McMaster University, Hamilton, ON, Canada

Abstract. We survey different approaches to evaluation of hydrodynamic forces in viscous incompressible flows focusing on techniques which do not explicitly require pressure information. A simple procedure is introduced which allows one to obtain a family of formulas involving only the velocity and vorticity fields by manipulating the Navier-Stokes equation. These formulas offer a number of computational advantages over standard techniques and also provide interesting physical insights about the relation between hydrodynamic forces and the dynamics of vortices in the flow. Finally, it is shown that a special treatment is required to evaluate hydrodynamic forces in steady flows in unbounded domains.

1 Introduction

Calculation of hydrodynamic forces acting on an object immersed in a fluid is one of the central objectives in many applied problems in fluid dynamics. In this contribution we will survey a range of different techniques for the evaluation of hydrodynamic forces which do not require the pressure information. Their derivation relies on suitable manipulation of the equations for the conservation of mass and momentum together with assumptions on the behavior of the velocity field at large distances from the obstacle. The resulting expressions are characterized by different degrees of computational efficiency and physical insight. We are concerned with incompressible flows in unbounded exterior domains (Figure 1(a)). In some derivations we will also consider truncations Ω_1 of the domain Ω obtained by imposing an exterior boundary Γ_1 (Figure 1(b)). We fix the origin of the coordinate system at the obstacle and assume that the obstacle remains motionless with the fluid velocity vanishing on its boundary. We also assume that there is a uniform flow $U_\infty \mathbf{e}_1$ at infinity ($\mathbf{e}_1$ is the unit vector corresponding to the OX axis). The fluid motion is governed by the Navier–Stokes system representing conservation of mass and momentum. This system of equations is assumed to have the following form:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\times\boldsymbol{\omega} + \boldsymbol{\nabla}\frac{\mathbf{u}^2}{2} + \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}\times\boldsymbol{\omega} = 0 \qquad \text{in } \Omega\times[0,T], \tag{1a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \qquad \text{in } \Omega\times[0,T], \tag{1b}$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \qquad \text{in } \Omega, \tag{1c}$$

$$\mathbf{u}\big|_{\Gamma_0} = 0 \qquad \text{in } [0,T], \tag{1d}$$

$$\mathbf{u} \longrightarrow U_\infty \mathbf{e}_1 \qquad \text{in } [0,T] \text{ for } |\mathbf{x}|\to\infty, \tag{1e}$$

where $\mathbf{u} = [u_1, u_2, u_3]$ is the velocity field, $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ is the vorticity, p is the pressure, ν represents the coefficient of the kinematic viscosity (the density of the fluid is assumed equal to unity), $\mathbf{u}_0$ is the initial condition, T represents the end of the time interval considered and $\mathbf{x} = [x_1, x_2, x_3]$ is the

[a] e-mail: bprotas@mcmaster.ca

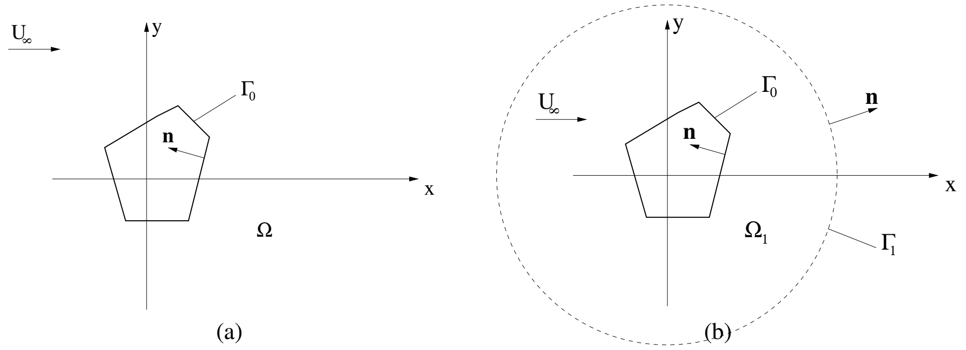

Fig. 1. Schematic of the flow past an obstacle Γ_0 in (a) an unbounded exterior domain Ω and (b) an exterior domain Ω_1 with an outer boundary Γ_1.

position vector. Given an object with a boundary Γ_0 characterized by the local unit normal vector $\mathbf{n}$ facing into the object (Figure 1a), the hydrodynamic force acting on this object is, by definition, given by the following expression

$$\mathbf{F} = \mathbf{F}^p + \mathbf{F}^\nu = \oint_{\Gamma_0} p\mathbf{n}\, d\sigma - \nu \oint_{\Gamma_0} \left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right]\mathbf{n}\, d\sigma = \oint_{\Gamma_0} p\mathbf{n}\, d\sigma + \nu \oint_{\Gamma_0} \mathbf{n} \times \boldsymbol{\omega}\, d\sigma. \tag{2}$$

The velocity gradient is defined as $[\boldsymbol{\nabla}\mathbf{u}]_{ij} = \partial u_i/\partial x_j$ and the two forms of the viscous term $\mathbf{F}^\nu$ are equivalent due to the identity $\oint_{\Gamma_0} (\boldsymbol{\nabla}\mathbf{u})^T \mathbf{n}\, d\sigma = 0$ valid for all incompressible fields $\mathbf{u}$. The arguments that we elaborate in this paper are valid in both two-dimensional (2D) and three-dimensional (3D) domains.

Application of the definition formula (2), which explicitly involves pressure p, is inconvenient in situations in which the flow field is characterized in terms of the velocity fields only (possibly together with their spatial derivatives). Such situations occur in experimental investigations in which techniques of the Particle Image Velocimetry (PIV) are used for measurements [1] and also in computational studies relying on the "non-primitive" formulation of the governing system (1), i.e., a formulation in which the pressure p is not explicitly present [2]. Provided that information about the velocity field is available in the entire domain Ω_1 and on its boundaries $\Gamma_0 \cup \Gamma_1$, cf. Figure 1(b), pressure can be recovered by solving the Poisson equation. However, this step is rather inconvenient and is usually avoided in practical situations. A number of techniques alternative to (2) have been proposed in the literature which allow one to evaluate hydrodynamic forces based on the velocity and vorticity fields alone. They are summarized in Section 2 below where we demonstrate how they can be derived as special cases using one general procedure and also discuss some of their advantages and disadvantages. The calculation of forces in the special case of steady flows in unbounded domains is then discussed briefly in Section 3. Conclusions are deferred to Section 4.

2 The Variational Formulation — A General Approach

In this section we show how a family of different approaches to the evaluation of hydrodynamic forces (2), which do not require the pressure information, can be derived by following a general procedure. It will depend on a vector-valued function $\boldsymbol{\gamma}$ and we will see how for different choices of this function formulas with quite distinct structure and computational properties will be obtained. In addition to the velocity and vorticity fields, $\mathbf{u}$ and $\boldsymbol{\omega}$, satisfying the governing system (1), in this section we will also assume that the velocity field approaches its far-field value (1e) sufficiently rapidly, i.e., $(\mathbf{u} - U_\infty \mathbf{e}_1)$ behaves as $O(|\mathbf{x}|^{-2})$ in 2D and as $O(|\mathbf{x}|^{-3})$ in 3D [3]. Below we will compute the component of the force acting in the direction given by a unit vector $\mathbf{a}$ (so that, upon choosing $\mathbf{a} = \mathbf{e}_1$, $\mathbf{F} \cdot \mathbf{a}$ will correspond to the drag force).

Procedure 1

1. *choose a function* $\boldsymbol{\gamma} \in [H^1(\Omega)]^D$*, where* $[H^1(\Omega)]^D$ *denotes the Sobolev space of vector–valued functions with square–integrable derivatives in* Ω*, such that*
$$\mathcal{B}\boldsymbol{\gamma}\big|_{\Gamma_0} = \mathcal{B}\mathbf{a}, \tag{3}$$
where $\mathcal{B} : \boldsymbol{\gamma}\big|_{\Gamma_0} \to \mathcal{B}\boldsymbol{\gamma}\big|_{\Gamma_0}$ *is a linear operator acting on the boundary values (traces) of the function* $\boldsymbol{\gamma}$*,*
2. *multiply the momentum equation* (1a) *by* $\boldsymbol{\gamma}$ *and integrate over the truncated domain* Ω_1
$$\int_{\Omega_1} \boldsymbol{\gamma} \cdot \left[\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} + \boldsymbol{\nabla}\frac{\mathbf{u}^2}{2}\right] d\Omega = \int_{\Omega_1} \boldsymbol{\gamma} \cdot \left[-\boldsymbol{\nabla} p - \nu \boldsymbol{\nabla} \times \boldsymbol{\omega}\right] d\Omega, \tag{4}$$
3. *use integration by parts and relation* (3) *valid on the boundary to extract from* (4) *the terms corresponding to* (2)*,*
4. *assume that* $\Gamma_1 \to \infty$ *which, given the assumptions on the behavior of* $\boldsymbol{\gamma}$ *and* $\mathbf{u}$ *for large* $|\mathbf{x}|$ *will remove the integrals defined on* Γ_1*.*

In order to obtain a unique function $\boldsymbol{\gamma}$, condition (3) has to be supplemented with an additional condition defined in the domain Ω. We will now illustrate how the above general procedure can lead, for different choices of this additional condition, and hence the function $\boldsymbol{\gamma}$, to the following well-known approaches.

2.1 The Impulse Formula in Unbounded Domains

The so-called "impulse formula" is obtained by choosing
$$\boldsymbol{\gamma} = \mathbf{a} \text{ in } \Omega, \text{ hence, by extension, } \mathcal{B} = \mathrm{Id} \;\Rightarrow\; \boldsymbol{\gamma}\big|_{\Gamma_0} = \mathbf{a}, \tag{5}$$
i.e., the function $\boldsymbol{\gamma}$ is constant and given by the vector $\mathbf{a}$ everywhere. Following our general procedure and using standard vector identities (see, e.g. [6]) we obtain
$$\mathbf{F} \cdot \mathbf{a} = -\frac{\mathbf{a}}{D-1} \cdot \frac{d}{dt} \int_{\Omega} \mathbf{x} \times \boldsymbol{\omega}\, d\Omega, \tag{6}$$
where $D = 2, 3$ is the spatial dimension. This relation was popularized by Saffman [4] and plays an important role in a number of theoretical considerations. While providing an interesting insight into the relationship between the force and vorticity dynamics, this approach has the disadvantage that integration is extended over the entire infinite domain. Consequently, vorticity at very large distances from the obstacle must be included which can be quite difficult in both numerical simulations and PIV measurements. In addition, the time derivative present in (6) tends to amplify noise.

2.2 The Impulse Formula in Bounded Domains

By proceeding as in Section 2.1, but abandoning Step 4 of Procedure 1, i.e., retaining a truncated domain Ω_1, we obtain a family of formulas of the type
$$\mathbf{F} = -\frac{1}{D-1}\frac{d}{dt}\int_{\Omega_1} \mathbf{x} \times \boldsymbol{\omega}\, d\Omega + \Big[\text{integral over } \Gamma_1\Big] + \Big[\text{integral over } \Gamma_0\Big], \tag{7}$$
where integration is restricted to the truncated domain Ω_1 and the far field contribution is contained in the integral over Γ_1. They were derived and analyzed by Noca et al. [5,6] and the reader is referred to the original papers for details. These formulas no longer require integration over an infinite domain, but still suffer from the presence of the time derivative. Furthermore, evaluation of the fluxes involved in the integrals over Γ_1 may be complicated.

2.3 The Quartapelle-Napolitano Approach

A fundamentally different approach, originally proposed in [7], is obtained by choosing the function $\boldsymbol{\gamma}$ in the form $\boldsymbol{\gamma} = -\boldsymbol{\nabla}\eta_a$, where η_a satisfies the following Neumann problem for the Laplace equation

$$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{\gamma} = -\Delta\eta_a = 0 & \text{in } \Omega, \\ \mathcal{B} = (\mathbf{n}, \cdot) & \Rightarrow \ (\mathbf{n}, \boldsymbol{\gamma}\big|_{\Gamma_0}) = -\mathbf{n} \cdot \boldsymbol{\nabla}\eta_a\big|_{\Gamma_0} = \mathbf{n} \cdot \mathbf{a}, \\ \boldsymbol{\gamma} \to 0 & \text{for } |\mathbf{x}| \to \infty \end{cases} \tag{8}$$

in which $(\cdot, \cdot)$ represents the standard Euclidean inner product. Following the steps of our general procedure and employing transformations described in detail in [12], we can express the pressure force as

$$\mathbf{F}^p \cdot \mathbf{a} = -\int_\Omega \boldsymbol{\nabla}\eta_a \cdot (\mathbf{u} \times \boldsymbol{\omega})\, d\Omega + \nu \oint_{\Gamma_0} \boldsymbol{\nabla}\eta_a \cdot (\mathbf{n} \times \boldsymbol{\omega})\, d\sigma. \tag{9}$$

The second term on the right–hand side in (9) is similar, but not equal, to the term representing the viscous stresses in (2). In order to obtain an expression for the total force, the viscous term $\mathbf{F}^\nu \cdot \mathbf{a}$ must be added to (9) resulting in the formula

$$F_a = \mathbf{F} \cdot \mathbf{a} = -\int_\Omega \boldsymbol{\nabla}\eta_a \cdot (\mathbf{u} \times \boldsymbol{\omega})\, d\Omega + \nu \oint_{\Gamma_0} (\boldsymbol{\nabla}\eta_a + \mathbf{a}) \cdot (\mathbf{n} \times \boldsymbol{\omega})\, d\sigma. \tag{10}$$

We remark that in the above expression the two terms involving the function η_a represent the contributions from the pressure force $\mathbf{F}^p \cdot \mathbf{a}$. Formula (10) has the advantage that, apart from the absence of the time–derivative, the integrand expression in the area integral includes a factor that rapidly decays with the distance from the obstacle. As a result, formula (10) is much more convenient to apply in numerical simulations where resolution of the velocity and vorticity fields is usually decreased far from the obstacle and appears also as a promising possibility for calculating forces based on PIV measurement data. This method has been further developed in different directions in [8–13]. It has been recognized, however, that the approach based on formula (10) has a certain shortcoming. We note that the expression for the pressure force $\mathbf{F}^p$ involves a boundary integral term proportional to the viscosity ν. In order to evaluate this term and the term representing viscous stresses, the distribution of vorticity *on the boundary* must be available which in many applications is rather inconvenient (in grid–based numerical methods and in PIV this may require the construction of complicated differentiation stencils). However, as was shown by the author in [14], formula (10) cannot be simplified by redefining the function $\boldsymbol{\gamma}$ in order to eliminate the boundary term. In this sense, it can be considered "optimal" within the family of approaches resulting from Procedure 1.

3 Calculation of Forces in Steady Flows in Unbounded Domains

In this section we briefly address the case of force computations in steady flows, characterized by the vanishing of the time derivative term $\frac{\partial \mathbf{u}}{\partial t}$ in (1a), in unbounded domains Ω. We observe that setting the time derivative in relation (6) to zero we obtain a somewhat surprising result that the hydrodynamic force $\mathbf{F}$ should vanish. This clearly contradicts the well-known empirical observations that the drag force (corresponding to $\mathbf{a} = \mathbf{e}_1$ in (6)) never vanishes in steady flows of fluids with finite viscosity $\nu > 0$. This apparent paradox was resolved by the author in [15] where it was demonstrated that relation (6) is not in fact valid for steady flows in unbounded domains. The reason is that such flows exhibit a *slower* (as compared to the corresponding unsteady flows) decay of the velocity field at infinity [16]. As a result, Step 4 of Procedure 1 fails, since some of the boundary integrals do not vanish when $\Gamma_1 \longrightarrow \infty$, rendering (6) incorrect. In such situations, one therefore needs to rely on formulation of the type (7) defined on finite domains Ω_1 which, due to the terms containing integrals over the contours Γ_0 and Γ_1, will predict finite drag even when the time derivative of the impulse integral vanishes.

4 Conclusions

In this contribution we surveyed a number of different approaches for the computation of hydrodynamic forces in flows of viscous incompressible fluids based on the velocity and vorticity fields only. We stressed how these different formulas can be derived by manipulating the governing equations via a single general procedure. We also highlighted the physical insights and computational advantages offered by the different formulas. It should be added that within the proposed framework it is also straightforward to account for the effects of the motion and/or deformation of the obstacle on the forces (the result will be the appearance of some additional integral terms defined on the obstacle boundary Γ_0). Likewise, expressions analogous to (6), (7) and (10) can be obtained for the hydrodynamic torque. Finally, we explained the reason why the calculation of forces in steady flows in unbounded domains, for which the impulse formula (6) is not applicable, requires special treatment.

References

1. J. Westerwee, G. E. Elsinga and R. J. Adrian, Ann. Rev. Fluid Mech. **45**, (2013), 409-436.
2. P. M. Gresho, Ann. Rev. Fluid Mech. **23**, (1991) 413–453.
3. R. Mizumachi, J. Math, Soc. Japan **36**, (1984) 497–522.
4. P. E. Saffman, *Vortex Dynamics* (Cambridge University Press, 1992).
5. F. Noca, D. Shiels and D. Jeon, J. Fluid Struc **11**, (1997) 345–350.
6. F. Noca, D. Shiels and D. Jeon, J. Fluid Struct **13**, (1999) 551–578.
7. L. Quartapelle and M. Napolitano, AIAA J. **21**, (1983) 911–913.
8. C.-C. Chang, Proc. Roy. Soc. A **437**, (1992) 517–525.
9. M. S. Howe, Q J Mech Appl. Math. **48**, (1995) 401–426.
10. C.-C. Chang and S.-Y. Lei, Proc. Roy. Soc. A **452**, (1996) 2369–2395.
11. C.-C. Chang, S.-Y. Su and S.-Y. Lei, Theor Comp. Fluid Dyn. **10**, (1998) 71–90.
12. B. Protas, A. Styczek and A. Nowakowski, J. Comput. Phys. **159**, (2000) 231–245.
13. L. S. Pan and T. T. Chew, J. Fluids Struct **16**, (2002) 71–82.
14. B. Protas, J. Fluid Struct **23**, (2007) 1207–1214.
15. B. Protas, J. Fluid Struct **27**, (2011) 1455–1460.
16. G. P. Galdi, *An Introduction to the Mathematical Theory of Navier–Stokes Equations, Vol. I: Linearized Steady Problems, Vol. II: Nonlinear Steady Problems* (Springer, 1994).