Assignment 5, Math 2S3
Due Apr. 7, in class

1. (a) Suppose that N is the $n \times n$ matrix with 1's above the diagonal and zeroes everywhere else. Compute e^N.

   (b) Suppose that $A = \lambda I + N$ - a Jordan block. Solve the system of linear differential equations $y' = Ay$ by using the result from the first part.

2. Prove that if V is a finite dimensional complex vector space with an hermitian product then it has an orthogonal basis.

3. Suppose that V is a K-vector space. Define a map from V to V^{**}, the dual of the dual space: for any $v \in V$, we need to define a linear functional $e(v)$ on the dual space. That is, if $\varphi \in V^*$, we need to say what $e(v)(\varphi)$ evaluates to. Let $e(v)(\varphi) = \varphi(v)$. Show that this e is a linear map and an injection; we are not assuming that V is finite dimensional.

4. Suppose that V is a finite dimensional complex vector space with a non-degenerate hermitian product. We say that an operator $A : V \to V$ is normal if $A^*A = AA^*$. Prove a spectral theorem for normal operators as follows:

   (a) First, show that if A is normal then A and A^* have a common eigenvector, v.

   (b) Now show that the orthogonal complement of v is invariant for both A and A^*.

   (c) Conclude by induction on the dimension of V that V has a basis of orthogonal eigenvectors for A.