Arts & Science 1D06 Quiz #4 SOLUTIONS

23 October 2015

Full Name:________________________ Student # :____________

TA:________________________

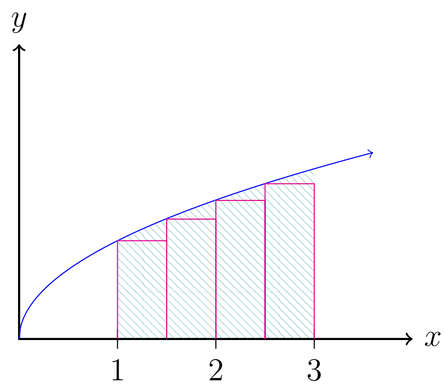
Please provide detailed solutions to the problems below. Correct responses without justification may not receive full credit. The use of a calculator is permitted.

[5 marks] (1) Find the approximate value of

$$\int_1^3 \sqrt{x}\, dx$$

using a Riemann sum with $n = 4$ rectangles and left endpoints.

Each rectangle has base $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$. The bottom-left corner of each rectangle has $x-$coordinate $x_i = \left(1 + \frac{1}{2}i\right)$, from $i = 0$ to $i = 3$. The area is therefore given by:



$$\begin{aligned}\text{Area} &= \sum_{i=0}^{n-1} f(x_i)\Delta x \\ &= \sum_{i=0}^{n-1} \left(1 + \frac{1}{2}i\right)^{\frac{1}{2}} \frac{1}{2} \\ &= \frac{1}{2}\left(1 + \sqrt{1.5} + \sqrt{2} + \sqrt{2.5}\right) \\ &= 2.61\end{aligned}$$

[1 marks] (2) Is the area calculated in (1) larger or smaller than the true area under the curve? Draw both the approximated and exact areas and show the difference between them.

The approximated area (i.e. the Riemann sum area) is given by the area within the rectangles, and the true area is given by the shaded region. Clearly the lattter is larger, since it includes the small pieces above each rectangle but below the graph.

[4 marks] (3) True or false: in general, if f is continuous on $[a, b]$ and $f'(x) > 0$ for $x \in (a, b)$, then:

$$\sum_{i=0}^{n-1} f(x_i^\star)\Delta x \le \int_a^b f(x)\, dx,$$

where $x_i^\star$ is the left endpoint of each interval. Explain.

True. If $f'(x) > 0$ then f is increasing. If we are using left endpoints to approximate the area, then the function has a larger value at the right side of the rectangle than it goes at the left, meaning the rectangle underestimates the area. This is true of all rectangles in the sum, so the Riemann sum is less than the true area.