MATH 3GR3 Assignment #4 Solutions
Due: Friday, 10 November, by 11:59pm

1. Let G be a group with $|G| < 300$. Suppose that G has a subgroup H with order 24 and a subgroup K with order 54. Determine the exact value of $|G|$.

   Solution: From Lagrange's theorem we know that both 24 and 54 are divisors of $|G|$. The only number less than 300 with this property is 216, the least common multiple of 24 and 54. Thus $|G| = 216$.

2. For each pair of groups G and H, determine if they are isomorphic:

   (a) $G = \mathbb{R}^*$, $H = \mathbb{C}^*$.

   Solution: They are not isomorphic. In the group $\mathbb{C}^*$, every element has a square root, i.e., for every $g \in \mathbb{C}^*$, there is some $h \in \mathbb{C}^*$ such that $hh = g$. This property does not hold in the group $\mathbb{R}^*$, and so the two groups cannot be isomorphic. In more detail, if $\phi : \mathbb{C}^* \to \mathbb{R}^*$ is an isomorphism then let $z \in \mathbb{C}^*$ with $\phi(z) = -1$. Let w be the square root of z in $\mathbb{C}^*$. Then $-1 = \phi(z) = \phi(ww) = \phi(w)^2$. This can't happen, since the square of any real number is non-negative.

   (b) $G = U(14)$, $H = U(18)$.

   Solution: Both of these groups are 6 element cyclic groups, and so are isomorphic (and isomorphic to $\mathbb{Z}_6$. To see that both are cyclic, it suffices to find elements of order 6 in each group. In $U(14)$, the element 3 has order 6 and in $U(18)$ the element 5 has order 6.

   (c) $G = \mathbb{Z}$, $H = \mathbb{R}$.

   Solution: These groups have different cardinalities, the first is countably infinite, while the second is uncountable. So, there is no bijection between the two groups and so they cannot be isomorphic. One can also prove this by noting that $\mathbb{Z}$ is cyclic, while $\mathbb{R}$ is not (for any non zero real number r, $\langle r\rangle$ is not equal to the set of all real numbers: since $\langle r\rangle = \{kr \mid k \in \mathbb{Z}\}$, then, for example, the real number $r/2 \notin \langle r\rangle$.).

(d) $G = \mathbb{Z}_{16}$ and $H = \mathbb{Z}_4 \times \mathbb{Z}_4$.

Solution: The first group is a cyclic group of order 16, while the second group is a non-cyclic group of order 16, so the two groups are not isomorphic. Each element of the second group has order at most 4 and so is not cyclic.

3. Show that the group $U(4) \times U(5)$ is isomorphic to the group $U(20)$.

Solution: It suffices to show that $U(20)$ contains two subgroups H and K that are isomorphic to $U(4)$ and $U(5)$ respectively and such that $U(20)$ is the internal direct product of H and K. Then by Theorem 9.27 we can conclude that $U(20) \cong H\times K \cong U(4)\times U(5)$. This makes use of the fact that if we have $G_1 \cong G_2$ and $G_3 \cong G_4$ then $G_1\times G_3 \cong G_2\times G_4$.

Since $U(4) = \{1, 3\}$ and $U(5) = \{1, 2, 3, 4\}$ then it can be seen that $U(4) \cong \mathbb{Z}_2$ and $U(5) \cong \mathbb{Z}_4$ (since the element 2 in $U(5)$ has order 4). So to find suitable H and K we need to look for elements of orders 2 and 4 that generate them. In $U(20)$ the element $h = 11$ has order 2 and the element $k = 3$ has order 4. If we set $H = \langle 11\rangle$ and $K = \langle 3\rangle$ then it can be seen that H and K satisfy the internal direct product conditions, and so we conclude that $U(20) \cong H \times K \cong U(4) \times U(5)$.

4. Let G_1 and G_2 be groups and suppose that H_1 is a subgroup of G_1 and H_2 is a subgroup of G_2. Show that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$. Find a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$ that is not of this form.

Solution: Let H_1 be a subgroup of G_1 and H_2 a subgroup of G_2. To show that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$ we need to show that it contains the identity element of $G_1 \times G_2$ and is closed under multiplication and taking inverses. Since both H_1 and H_2 are subgroups of G_1 and G_2 respectively, then e_{G_1} is in H_1 and e_{G_2} is in H_2 and so (e_{G_1}, e_{G_2}), the identity element of $G_1\times G_2$, is in $H_1\times H_2$. If (h_1, h_2) and (h_1', h_2') are in $H_1 \times H_2$ then $(h_1, h_2)(h_1', h_2') = (h_1h_1', h_2h_2') \in H_1 \times H_2$

since both H_1 and H_2 are closed under multiplication. Thus $H_1 \times H_2$ is also closed under multiplication. The inverse of (h_1, h_2) is the element (h_1^{-1}, h_2^{-1}), which is in $H_1 \times H_2$ since $h_1^{-1} \in H_1$ and $h_2^{-1} \in H_2$, and so $H_1 \times H_2$ is closed under taking inverses. Thus $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

The set $H = \{(0,0), (2,1)\}$ is a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$, since it contains the identity element of that group and is closed under addition and the taking of inverses. H is not of the form $H_1 \times H_2$ for some subgroups H_1 of $\mathbb{Z}_4$ and H_2 of $\mathbb{Z}_2$ since the only subgroups of this form of order 2 are $\{0,2\} \times \{0\}$ and $\{0\} \times \{0,1\}$.

Alternatively, any subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$ that is of the form $H_1 \times H_2$ for some subgroups H_1 of $\mathbb{Z}_4$ and H_2 of $\mathbb{Z}_2$ and that contains $(2,1)$ would also have to contain $(2,0)$ and $(0,1)$. So the given subgroup H is not of this form.

5. Let G be a group, H a subgroup of G, and $g \in G$.

   (a) Show that the map $f : G \to G$ defined by $f(x) = gxg^{-1}$ is an isomorphism from G to G.

   Solution: f is one-to-one since if $f(x) = f(y)$ then $gxg^{-1} = gyg^{-1}$. After cancelling g and g^{-1}, we get that $x = y$. f is onto, since if $y \in G$ then $f(g^{-1}yg) = y$. For $x, y \in G$, $f(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = f(x)f(y)$. Thus, f is an isomorphism.

   (b) Show that the set $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G, if H is a subgroup of G.

   Solution: $e \in gHg^{-1}$ since $e \in H$ and $e = geg^{-1}$. If $x, y \in gHg^{-1}$ then $x = gug^{-1}$ and $y = gvg^{-1}$ for some $u, v \in H$. Then $xy = gug^{-1}gvg^{-1} = g(uv)g^{-1} \in gHg^{-1}$. Also, $x^{-1} = (gug^{-1})^{-1} = gu^{-1}g^{-1} \in gHg^{-1}$. Thus gHg^{-1} is a subgroup of G.

   (c) Show that $\bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G if H is a subgroup of G.

   Solution: Let $K = \bigcap_{g \in G} gHg^{-1}$. Since the intersection of a collection of subgroups of G is also a subgroup of G, then by part (b) it follows that K is a subgroup of G. To show that K is a normal subgroup of G, it suffices to show that for all $a \in G$, $aKa^{-1} \subseteq K$. By the definition of K, this amounts to showing that if $k \in K$ and $g \in G$, then the element $aka^{-1} \in gHg^{-1}$. Since $k \in K$, then in particular, $k \in uHu^{-1}$, where $u = a^{-1}g$ and so $k = uyu^{-1}$ for some $y \in H$. Then $aka^{-1} = auyu^{-1}a^{-1} = gyg^{-1} \in gHg^{-1}$, as required.

6. Show that every group of order 4 is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution: Let G be a group of order 4. If G contains an element of order 4 then it is cyclic, and so is isomorphic to the cyclic group $\mathbb{Z}_4$. If G does not have an element of order 4, then by Lagrange's theorem, all of the non-identity elements of G have order 2. So $G = \{e, a, b, c\}$ for some elements a, b, and c of order 2. If one fills in the Cayley table for G, we are forced to conclude that $ab = c = ba$, $ac = b = ca$, and $bc = a = cb$. These equalities, along with the fact that e is the identity element and the other elements have order 2, completely determine the Cayley table for G. It can now be checked that the map from G to $\mathbb{Z}_2 \times \mathbb{Z}_2$ that maps e to $(0, 0)$, a to $(0, 1)$, b to $(1, 0)$ and c to $(1, 1)$ is an isomorphism.

One can also solve this question by referring to the solution to question #3 of Assignment #1. In that question, it is shown that up to a rearrangement of elements (i.e., up to isomorphism), there are exactly two distinct Cayley tables for a group of order 4. One of these tables contains an element of order 4, and so such a group is isomorphic to $\mathbb{Z}_4$, while the other one can be seen to be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (by elimination, it has to be isomorphic to this group, since it is not cyclic).

7. (a) Show that $H = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a normal subgroup of A_4.

Solution: Since $|A_4| = 12$ and $|H| = 4$ then there are three left and three right cosets of H in A_4. They are: H,

$$(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3\ 4), (2\ 4\ 3), (1\ 4\ 2)\} = H(1\ 2\ 3),$$

and

$$(1\ 3\ 2)H = \{(1\ 3\ 2), (2\ 3\ 4), (1\ 2\ 4), (1\ 4\ 3)\} = H(1\ 3\ 2).$$

So all left cosets are right cosets.

(b) Show that H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution: From the previous question, we know that H is isomorphic to one of $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, since H is a four element group. Since all of the elements of H have order 1 or 2, it follows that H is not cyclic and so must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) Produce the Cayley table of the quotient group A_4/H.

Solution: The quotient has size 3 and so is isomorphic to the cyclic group $\mathbb{Z}_3$. Here is its Cayley table:

$\cdot$	H	$(1\ 2\ 3)H$	$(1\ 3\ 2)H$
H	H	$(1\ 2\ 3)H$	$(1\ 3\ 2)H$
$(1\ 2\ 3)H$	$(1\ 2\ 3)H$	$(1\ 3\ 2)H$	H
$(1\ 3\ 2)H$	$(1\ 3\ 2)H$	H	$(1\ 2\ 3)H$

(d) Show that H is the only non-trivial normal subgroup of A_4.

Solution: We've seen that A_4 does not have a subgroup of order 6 and so by Lagrange's theorem, we need only consider subroups of size 2, 3, or 4. If $K \leq A_4$ and $|K| = 2$, then $K = \{id, \sigma\}$ for some element $\sigma \in A_4$ of order 2. There are 3 possibilities for σ, and in each case, it can be shown that K is not normal. For example, if $\sigma = (1\ 2)(3\ 4)$, then the left coset $(1\ 2\ 3)K = \{(1\ 2\ 3), (1\ 3\ 4)\}$, while the right coset $K(1\ 2\ 3) = \{(1\ 2\ 3), (2\ 4\ 3)\}$. If $|K| = 3$ then K is generated by an element τ of A_4 of order 3. There are 8 possibilities for τ, and in each case, it can be shown that K is not normal. For example, if $\tau = (1\,2\,3)$, then $K = \{id, (1\,2\,3), (1\,3\,2)\}$. The left coset $(2\ 3\ 4)K = \{(2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 2)\}$ while the right coset $K(2\ 3\ 4) = \{(2\ 3\ 4), (1\ 2)(3\ 4), (1\ 3\ 4)\}$.

8. Let G be a group and define $Z(G)$ to be the subset $\{g \in G\ :\ gx = xg \text{ for all } x \in G\}$.

(a) Show that $Z(G)$ is a normal subgroup of G.

Solution: $Z(G)$ contains the identity element, since it commutes with every element of G. If a, $b \in Z(G)$, and if $g \in G$ then

$$(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab).$$

This shows that $Z(G)$ is closed under products.To show that $a^{-1} \in Z(G)$, we note that $ag = ga$ implies that $g^{-1}a = ag^{-1}$, by cancellation. From this it follows that $a^{-1} \in Z(G)$. Thus $Z(G)$ is a subgroup of G.

To show that $Z(G)$ is normal, it suffices to show that if $g \in G$ then $gZ(G)g^{-1} \subseteq Z(G)$. If $z \in Z(G)$, then $gzg^{-1} = zgg^{-1} = z \in Z(G)$, since z commutes with all elements from G, and in particular with the element g.

(b) Compute $Z(D_4)$.

Solution: Note, this is a simpler version of question #3 (b) from the previous assignment. The only elements of

$$D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

that commute with all other elements of D_4 are e and r^2. So, $Z(D_4) = \{e, r^2\}$. To see this, we can use that $r^4 = s^2 = e$ and that $srs = r^{-1}$. It follows that $r^2r^i = r^{2+i} = r^{i+2} = r^ir^2$ for any i, and

$$r^2(sr^i) = r(rs)r^i = r(sr^{-1})r^i = (rs)r^{i-1} = (sr^{-1})r^{i-1} = sr^{i-2},$$

while $(sr^i)r^2 = sr^{i+2}$. Since $i-2$ and $i+2$ differ by exactly 4, then in D_4, $r^{i-2} = r^{i+2}$ for any i. So $r^2 \in Z(D_4)$.

To see that there are no other elements in $Z(D_4)$, we need to show that every other element of D_4 fails to commute with all elements. The following inequalities demonstrate this: $sr = r^{-1}s \neq rs$, $(r^3)(sr) = r^2s \neq s = (sr)(r^3)$, and $(sr^3)(sr^2) = s(r^3r^2)s = srs = r^{-1} \neq r = (sr^2)(sr^3)$.

(c) Compute the Cayley table for the quotient group $D_4/Z(D_4)$.

Solution: Since $|D_4| = 8$ and $|Z(D_4)| = 2$, it follows that there are four left cosets of $Z(D_4)$ in D_4 and that $|D_4/Z(D_4)| = 4$. From an earlier question, it follows that $D_4/Z(D_4)$ is isomorphic to one of $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The four left cosets of $Z = Z(D_4)$ are: $Z = \{e, r^2\}$, $rZ = \{r, r^3\}$, $sZ = \{s, sr^2\}$, and $srZ = \{sr, sr^3\}$ and the Cayley table is:

$\cdot$	Z	rZ	sZ	srZ
Z	Z	rZ	sZ	srZ
rZ	rZ	Z	srZ	sZ
sZ	sZ	srZ	Z	rZ
srZ	srZ	sZ	rZ	Z

Since every element in this quotient has order 1 or 2, it follows that it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

9. Let G be a group and let $D = \{(g, g) : g \in G\}$, a subgroup of $G \times G$.

(a) Show that D is isomorphic to the group G.

Solution:
Let $f : G \to D$ be defined by $f(g) = (g, g)$. Then f is a bijection since it is clearly one-to-one and onto. It is an isomorphism since $f(gh) = (gh, gh) = (g, g)(h, h) = f(g)f(h)$.

(b) Show that D is a normal subgroup of the group $G \times G$ if and only if G is an abelian group.

Solution: First, D is a subgroup of $G \times G$ since it is the image of the isomorphism from G to $G \times G$ that maps g to (g, g). This can also be checked by showing that D satisfies the conditions for being a subgroup.
If G is abelian, then so is $G \times G$ (this can be checked, using the definition of the group operation on $G \times G$). Since every subgroup of an abelian group is normal, it follows that D is a normal subgroup of $G \times G$. Conversely, suppose that D is a normal

subgroup and let $a,\ b \in G$. Since D is normal, then the element $(a,b)(a,a)(a,b)^{-1}$ is a member of D, since (a,a) is. But

$$(a,b)(a,a)(a,b)^{-1} = (a,b)(a,a)(a^{-1},b^{-1}) = (aaa^{-1},bab^{-1}) = (a,bab^{-1}).$$

Since elements of D are of the form (x,x) then $a = bab^{-1}$ and so $ab = ba$. This shows that G is abelian if D is a normal subgroup.

10. Read over the SageMath tutorials at the ends of Chapters 6 and 9 and perform the following calculations. To submit your calculations, either take a screenshot (or maybe a picture) of the webpage that contains them or include a copy of the link (i.e., URL) that is produced by the SageCell "share" button, or just copy and paste your commands and the results into the document that you upload to Avenue to Learn.

    (a) Compute $\phi(2023)$, where ϕ is the Euler ϕ-function.

    (b) Produce the cyclic group G of order 40 using the "CyclicPermutationGroup" function and the group D_{20} and then use the "is_isomorphic" function to determine if these two groups are isomorphic.

    Solution: For the SageMath question, click on the following link to see a solution: link

Supplementary problems from the textbook (not to be handed in)

- From Chapter 9, questions 16, 19, 22, 23, 48, 52
- From Chapter 10, questions 1, 2, 4, 5, 6, 9, 13