MATH 3GR3 Midterm Test #2 Solutions

Midterm Test
Duration of test: 50 minutes
McMaster University
November 14, 2023

Instructor: Matt Valeriote

Last Name:_____________________________ First Name:_______________

Student Number:_____________________________________

Please answer all five questions. To receive full credit, provide justifications for your solutions. For all questions, write your answers in the answer booklet that has been provided. Please be sure to include your name and student number on all sheets of paper that you hand in.

NOTE: In your solutions you may make use of any theorems or results discussed in the lectures. You may not use other theorems or results, unless you fully justify them. This includes results from the homework assignments.

No aids are allowed.

The number of points each question is worth is indicated in the margin; the maximal number of marks is 25.

Score

Question	1	2	3	4	5	Total
Score						

[6] 1. (a) State Lagrange's Theorem.

Solution: See textbook.

(b) Find a subgroup of S_4 that is isomorphic to the group $\mathbb{Z}_4$.

Solution: According to Cayley's Theorem, such a subgroup should exist. In this case, since $\mathbb{Z}_4$ is cyclic of order 4, then we need to find an element σ of S_4 that has order 4. Then $\langle \sigma \rangle$ is a subgroup of S_4 that is isomorphic to $\mathbb{Z}_4$. This is because any two cyclic groups of the same order are isomorphic.

(c) Does S_4 have an element of order 5? Justify your answer to receive credit.

Solution: The order of S_4 is $4! = 24$. Since 5 does not divide 24, then by Lagrange's Theorem, S_4 does not have an element of order 5.

(d) Does S_4 have an element of order 6? Justify your answer to receive credit.

Solution: The order of an element of S_4 can be determined by first expressing it as a product of disjoint cycles. Its order will be the least common multiple of the lengths of each of these factors. In the case of S_4, the possible ways of expressing a member as a product of disjoint cycles are: 2-cycle, 3-cycle, 4-cycle, and the product of 2 disjoint 2-cycles. Since the orders of elements that can be expressed in this manner are 2, 3, and 4, then we conclude that S_4 does not have an element of order 6.

[4] 2. Find an element of the group $\mathbb{Z}_6 \times \mathbb{Z}_4$ of order 12. Does this group have an element of order 8? Justify your answers to receive credit.

Solution: The element $(2, 1)$ has order 12, since $|(2, 1)| = \text{lcm}(|2|, |1|) = \text{lcm}(3, 4) = 12$.

There is no element of order 8, since for any $a \in \mathbb{Z}_6$ and $b \in \mathbb{Z}_4$, $|(a, b)| = \text{lcm}(|a|, |b|)$, and since $|a|$ divides 6 and $|b|$ divides 4, $\text{lcm}(|a|, |b|) \neq 8$.

3. Determine which of the following pairs of groups are isomorphic. Justify
[6] your answers to receive credit.

(a) $\mathbb{Z}_4$ and $U(8)$.

Solution: It can be checked that all of the elements of $U(8) = \{1, 3, 5, 7\}$ have order 1 or 2 and so $U(8)$ cannot be isomorphic to $\mathbb{Z}_4$, since $\mathbb{Z}_4$ has an element of order 4.

(b) A_4 and $\mathbb{Z}_{12}$.

Solution: A_4 is not abelian, while $\mathbb{Z}_{12}$ is, so these groups are not isomorphic. Also, A_4 is not cycli, while $\mathbb{Z}_{12}$ is.

(c) $\mathbb{Z}_6 \times \mathbb{Z}_7$ and $\mathbb{Z}_{21} \times \mathbb{Z}_2$.

order

Solution: These groups are both isomorphic to $\mathbb{Z}_{42}$, since each of them contains an element of 42. Since all cyclic groups of a given order are isomorphic, it follows that they are isomorphic. Alternatively, we proved a theorem that states that if m and n are relatively prime, then $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$.

[4] 4. (a) List the left cosets of the subgroup $\langle(1, 2)\rangle$ in the group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Solution: $N = \langle(1, 2)\rangle = \{(0, 0), (1, 2), (2, 1)\}$ is one of the left cosets of this subgroup. Since $|\mathbb{Z}_3 \times \mathbb{Z}_3| = 9$ and the subgroup has order 3, then there are exactly 3 left cosets of N. Since $(0, 1) \notin N$, then a second left coset is $(0, 1) + N = \{(0, 1), (1, 0), (2, 2)\}$. The remaining 3 elements of the group must form the third left coset: $(0, 2) + N = \{(1, 1), (2, 0), (0, 2)\}$.

(b) Produce the Cayley table of the factor group $(\mathbb{Z}_3 \times \mathbb{Z}_3)/\langle(1, 2)\rangle$.

Solution: The Cayley table for the factor group $\mathbb{Z}_3 \times \mathbb{Z}_3/N$ is

$+$	N	$(0,1)+N$	$(0,2)+N$
N	N	$(0,1)+N$	$(0,2)+N$
$(0,1)+N$	$(0,1)+N$	$(0,2)+N$	N
$(0,2)+N$	$(0,2)+N$	N	$(0,1)+N$

5. For each of the following statements, decide if they are **True** or **False**. To receive credit for your answers, you must justify them, by either providing a proof, if the statement is True, or a counter example, if the statement is False. [5]

(a) If the group G is abelian and N is a normal subgroup of G, then the factor group G/N is abelian.

Solution: This is **True**. If $C_1,\ C_2 \in G/N$, then $C_1 = aN$ and $C_2 = bN$ for some $a,\ b \in G$. Then

$$C_1 C_2 = (aN)(bN) = (ab)N = (ba)N = (bN)(aN) = C_2 C_1.$$

So, G/N is abelian.

(b) If N is a normal subgroup of the group G such that both N and G/N are abelian, then the group G is abelian.

Solution: This is **False**. For example, let $G = S_3$ and $N = \langle (1,2,3) \rangle = \{id, (1,2,3), (1,3,2)\}$. Then N is a normal subgroup of S_3 since it has index 2. We have that S_3 is not abelian, but both N and G/N are.