MATH 4L03/6L03 Assignment #1 Solutions

Due: Monday, September 16, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

Important note: All work submitted for grading must be your own. You may discuss homework problems and related material with other students, but you must not submit work copied from others. You may not use any generative Artificial Intelligence system, such as ChatGPT, to assist you in preparing your solutions to the homework problems.

1. There is an island in which every occupant is either a knight or a knave. Knights always tell the truth and knaves always lie. By formalizing the following statements within propositional logic and using the method of truth tables, answer the following puzzles:

   a) Suppose that Alice and Bob are occupants of this island and Alice says "At least one of us is a knave". What are Alice and Bob? [Hint: Let the propositional variable A stand for the statement "Alice is a knight" and let B stand for the statement "Bob is a knight". Then the statement uttered by Alice can be formalized by the wff $(\neg A \vee \neg B)$ and under the above interpretations of A and B, the formula $A \leftrightarrow (\neg A \vee \neg B)$ is true. Now find all truth assignments on $\{A, B\}$ that makes this formula true. There should be exactly one such assignment if this puzzle has a unique answer.]

   b) Now suppose that Alice says "Either I am a knave or Bob is a knight". What are Alice and Bob?

   c) A third person Eve enters the picture, Alice says "All of us are knaves" and Bob says "Exactly one of us is a knave". What are Alice, Bob and Eve?

   d) Now Alice says "Bob is a knave" and Bob says "Alice and Eve are of the same type" (I think that Bob means that either Alice and Eve are both knights or they are both knaves). What is Eve?

   These are taken from the book "What is the name of this book?" by the logician Raymond Smullyan.

   Solution:

a) Using the hint, the formula $A \leftrightarrow (\neg A \vee \neg B)$ is true. The only way to satisfy this formula is to make A true and B false, so A is a knight and B is a knave.

b) The statement $A \leftrightarrow (\neg A \vee B)$ is true and the only way to satisfy it is to set A and B true. Thus both A an B are knights.

c) The statements $A \leftrightarrow \neg(A \vee B \vee C)$ and $B \leftrightarrow ((\neg A \wedge B \wedge C) \vee (A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C))$ are true. There are two ways to make both of these statements true at the same time: set A false, C true and B either true or false. So, A is a knave, C is a knight and the type of B is undetermined.

d) The statements $A \leftrightarrow \neg B$ and $B \leftrightarrow (A \leftrightarrow C)$ are both true. The only way to satisfy both of them is to set A true and B false, or A false and B true, and C false. Thus, C is a knave.

2. Decide which of the following strings are formulas. Justify your answers by using parse trees. For each string that is a formula, list all of their subformulas.

   (a) $((p \vee q) \wedge r)$,
   (b) $(\neg p \vee \neg\neg q))$,
   (c) $(p \vee q \wedge r)$,
   (d) $(((p \to q) \to r) \to s)$,
   (e) $((p_1 \to (\neg p_2)) \leftrightarrow p_3)$.

   Solution: The strings from parts (a) and (d) are formulas. The other three are not. For (b), the number of left and right brackets differ, and so, by a theorem from the lectures, this string is not a formula. For (c), there is a missing pair of brackets. As it stands, it is not clear which connective is the principal one, i.e., the one applied in order the last step of forming this string. For (e), there is an extraneous set of brackets around $\neg p_2$; without them, the string is a formula.

3. (a) Prove that the number of subformulas of a formula ϕ is at most $2|\phi| + 1$. Recall that $|\phi|$ is equal to the number of occurrences of connectives in ϕ.

Solution: We prove this by induction on the length of ϕ. For the base case, suppose that $|\phi| = 0$. Then $\phi = p$ for some propositional variable p. In this case, the only subformula of ϕ is ϕ itself and so the claim holds for this ϕ.

Now suppose that the claim holds for all formulas of length at most n and suppose that $|\phi| = n + 1$. There are several cases to consider.

- $\phi = \neg\theta$ for some formula θ. Then $|\theta| = n$ and so the claim holds for θ. Thus the number of subformulas of θ is at most $2n + 1$. By definition, the subformulas of ϕ consist of the subformulas of θ, along with ϕ itself. So the number of subformulas of ϕ is one more than the number for θ. Thus the number of subformulas of ϕ is at most $2n + 1 + 1$. Since $2n+2 \leq 2(n+1)+1$ for all n, the claim has been established in this case.
- The remaining cases are all similar, and so we will only provide details for one of them. Suppose that $\phi = (\theta \to \gamma)$ for formulas θ and γ. Then $|\theta| + |\gamma| = n$ and so by assumption, the claim holds for these formulas. Let $|\theta| = m$ and $|\gamma| = k$. Then the number of subformulas of θ is at most $2m + 1$ and the number of subformulas of γ is at most $2k + 1$. In this case, the subformulas of ϕ consist of the subformulas of θ and γ, along with ϕ and so the number of subformulas of ϕ is at most

  $$(2m+1)+(2k+1)+1 = 2(m+k)+3 = 2n+3 = 2(n+1)+1,$$

  as required.

(b) Prove that a subformula of a subformula of a formula ϕ is also a subformula of ϕ.

Solution: We prove this by induction on $|\phi|$. In the case where $|\phi| = 0$, the only subformula of ϕ is ϕ and the claim follows. Suppose that it holds for all formulas of length at most n and suppose that $|\phi| = n + 1$. Let α be a subformula of ϕ and β a subformula of α. We need to show that β is a subformula of ϕ.

There are several cases to consider.

- $\phi = \neg\theta$ for some formula θ. Then $|\theta| = n$ and so the claim holds for θ. Either $\alpha = \phi$ or α is a subformula of θ. In the former case, it follows that β is a subformula of ϕ, since it is a subformula of α and $\alpha = \phi$. In the latter case, it follows by induction that β is a subformula of θ. But by definition, every subformula of θ is a subformula of ϕ and so β is also a subformula of ϕ.
- The remaining cases are all similar, and so we will only provide details for one of them. Suppose that $\phi = (\theta \vee \gamma)$ for formulas θ and γ. Then $|\theta|$, $|\gamma| len$ and so by assumption, the claim holds for these formulas. Since α is a subformula of ϕ then as in the previous case, it is either equal to ϕ or is a subformula of one of θ or γ. In the former case it follows that β is a subformula of ϕ, since it is a subformula of α and $\alpha = \phi$. In the latter case, it follows by induction that β is a subformula of θ ot of γ. But by definition, every subformula of θ or γ is a subformula of ϕ and so β is also a subformula of ϕ.

4. Prove that if ϕ is a formula whose only logical connectives come from the set $\{\wedge, \vee\}$ then ϕ is satisfiable.

   Solution: Let $\phi \in Form(P, S)$ where S includes the connectives $\wedge$ and $\vee$. Let $\nu : Form(P, S) \to \{T, F\}$ be the truth assignment with $\nu(p) = T$ for all $p \in P$. We will show by induction on the length of ϕ that if the only connectives in ϕ come from $\{\wedge, \vee\}$ then $\nu(\phi) = T$, thereby showing that such formulas are always satisfiable.

   For the base case, when the length of ϕ is 0, then $\phi = p$ for some $p \in P$ and so $\nu(\phi) = T$. For the induction step, we have $\phi = (\theta \wedge \psi)$ or $(\theta \vee \psi)$ for some shorter formulas θ and ψ whose connectives (if any) must come from $\{\wedge, \vee\}$. By induction $\nu(\theta) = \nu(\psi) = T$. From this it follows that $\nu(\phi) = T$ as required.

5. Which of the following formulas are tautologies? satisfiable? Below, p, q, r and s can be taken as propositional variables. Note that to improve readability other sorts of brackets are used in the formulas. Treat them as regular left and right parentheses. Also, when clear, some pairs of brackets have not been included in formulas.

a) $[(p \vee q) \to (p \wedge r)] \wedge [(q \to r) \vee q]$.

b) $(p \to (r \to s)) \to ((p \wedge r) \to s)$.

c) $[\neg p \wedge (\{[\neg\neg p \wedge (p \wedge q)] \wedge (q \to p)\} \to p)] \to q$.

Solution: The formula in a) is satisfiable, but is not a tautology. The formula in b) is a tautology, and the formula in c) is satisfiable, but is not a tautology. In a), the truth assignment that sets all propositional variables to T satisfies the formula, while the assignment that sets p to F, q to T, and r to T falsifies the formula.

To see that b) is a tautology, just produce its truth table, or argue that it cannot be falsified as follows: for ν to make this implication have truth value F, we must have $\nu((p \to (r \to s))) = T$ and $\nu(((p \wedge r) \to s)) = F$. For the latter to occur, we must have $\nu(p) = \nu(r) = T$ and $\nu(s) = F$. But with this ν we have $\nu((p \to (r \to s))) = F$ so this formula cannot be falsified.

For c), any assignment that sets q to be T will satisfy the formula (since $(\theta \to \psi)$ is always true when ψ is true). The assignment that set p to be F and q to be F falsifies the formula.

6. Show that the formulas $\neg p \vee q$ and $p \to q$ are logically equivalent. This shows that the connective $\to$ is definable (in some sense) from the connectives $\neg$ and $\vee$. Show that $\vee$ is definable in this way from the connective $\to$. In fact, we could develop propositional logic just using the two connectives $\neg$ and $\vee$. The resulting language is as expressive as the one that includes all of the usual connectives.

   Solution: $\nu(\neg p \vee q) = F$ if and only if $\nu(p) = T$ and $\nu(q) = F$ if and only if $\nu(p \to q) = F$, so these formulas are logically equivalent. One can also show this using truth tables. Similarly one can show that $p \vee q$ is logically equivalent to $((p \to q) \to q)$.

7. Show that the following pairs of formulas are logically equivalent. Justify your answers.

   (a) $((p \to q) \wedge (q \to p))$ and $((p \wedge q) \vee (\neg p \wedge \neg q))$.

   (b) $((p \leftrightarrow q) \leftrightarrow r)$ and $(p \leftrightarrow (q \leftrightarrow r))$.

Solution: You can establish these equivalences using truth tables. Alternatively, for (a), we see that $\nu(((p \to q) \wedge (q \to p))) = T$ if and only if $\nu(p) = \nu(q)$ if and only if $\nu((p \wedge q)) = T$ or $\nu((\neg p \wedge \neg q)) = T$ if and only if $\nu(((p \wedge q) \vee (\neg p \wedge \neg q))) = T$.

For (b), both formulas evaluate to T if and only if all three variables are assigned the value T or exactly one of them is assigned the value T. So, these formulas are logically equivalent.

8. Consider the following 3-variable logical connective $T(\alpha, \beta, \gamma)$ that is true only when exactly two of the three formulas α, β, and γ are true. Find a formula ϕ that just uses the usual connectives from $\{\neg, \wedge, \vee, \to, \leftrightarrow\}$ and the three propositional variables p, q, and r that is logically equivalent to the formula $T(p, q, r)$. Justify your answer by showing that the truth tables for $T(p, q, r)$ and ϕ are the same.

   Solution: The following (long) formula works. There may be a more concise way to express this 3-variable connective:

$$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r).$$