MATH 4L03/6L03 Assignment #1

Due: Monday, September 16, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

Important note: All work submitted for grading must be your own. You may discuss homework problems and related material with other students, but you must not submit work copied from others. You may not use any generative Artificial Intelligence system, such as ChatGPT, to assist you in preparing your solutions to the homework problems.

1. There is an island in which every occupant is either a knight or a knave. Knights always tell the truth and knaves always lie. By formalizing the following statements within propositional logic and using the method of truth tables, answer the following puzzles:

In some parts of this question, it might not be possible to determine whether an individual must be a knight or must be a knave.

a) Suppose that Alice and Bob are occupants of this island and Alice says "At least one of us is a knave". What are Alice and Bob? [Hint: Let the propositional variable A stand for the statement "Alice is a knight" and let B stand for the statement "Bob is a knight". Then the statement uttered by Alice can be formalized by the wff $(\neg A \vee \neg B)$ and under the above interpretations of A and B, the formula $A \leftrightarrow (\neg A \vee \neg B)$ is true. Now find all truth assignments on $\{A, B\}$ that makes this formula true. There should be exactly one such assignment if this puzzle has a unique answer.]

b) Now suppose that Alice says "Either I am a knave or Bob is a knight". What are Alice and Bob?

c) A third person Eve enters the picture, Alice says "All of us are knaves" and Bob says "Exactly one of us is a knave". What are Alice, Bob and Eve?

d) Now Alice says "Bob is a knave" and Bob says "Alice and Eve are of the same type" (I think that Bob means that either Alice and Eve are both knights or they are both knaves). What is Eve?

These are taken from the book "What is the name of this book?" by the logician Raymond Smullyan.

2. Decide which of the following strings are formulas. Justify your answers by using parse trees. For each string that is a formula, list all of their subformulas.

   (a) $((p \vee q) \wedge r)$,

   (b) $(\neg p \vee \neg\neg q))$,

   (c) $(p \vee q \wedge r)$,

   (d) $(((p \to q) \to r) \to s)$,

   (e) $((p_1 \to (\neg p_2)) \leftrightarrow p_3)$.

3. (a) Prove that the number of subformulas of a formula ϕ is at most $2|\phi| + 1$. Recall that $|\phi|$ is equal to the number of occurrences of connectives in ϕ.

   (b) Prove that a subformula of a subformula of a formula ϕ is also a subformula of ϕ.

4. Prove that if ϕ is a formula whose only logical connectives come from the set $\{\wedge, \vee\}$ then ϕ is satisfiable.

5. Which of the following formulas are tautologies? satisfiable? Below, p, q, r and s can be taken as propositional variables. Note that to improve readability other sorts of brackets are used in the formulas. Treat them as regular left and right parentheses. Also, when clear, some pairs of brackets have not been included in formulas.

   a) $[(p \vee q) \to (p \wedge r)] \wedge [(q \to r) \vee q]$.

   b) $(p \to (r \to s)) \to ((p \wedge r) \to s)$.

   c) $[\neg p \wedge (\{[\neg\neg p \wedge (p \wedge q)] \wedge (q \to p)\} \to p)] \to q$.

6. Show that the formulas $\neg p \vee q$ and $p \to q$ are logically equivalent. This shows that the connective $\to$ is definable (in some sense) from the connectives $\neg$ and $\vee$. Show that $\vee$ is definable in this way from the connective $\to$. In fact, we could develop propositional logic just using the two connectives $\neg$ and $\vee$. The resulting language is as expressive as the one that includes all of the usual connectives.

7. Show that the following pairs of formulas are logically equivalent. Justify your answers.

(a) $((p \to q) \land (q \to p))$ and $((p \land q) \lor (\neg p \land \neg q))$.

(b) $((p \leftrightarrow q) \leftrightarrow r)$ and $(p \leftrightarrow (q \leftrightarrow r))$.

8. Consider the following 3-variable logical connective $T(\alpha, \beta, \gamma)$ that is true only when exactly two of the three formulas α, β, and γ are true. Find a formula ϕ that just uses the usual connectives from $\{\neg, \land, \lor, \to, \leftrightarrow\}$ and the three propositional variables p, q, and r that is logically equivalent to the formula $T(p, q, r)$. Justify your answer by showing that the truth tables for $T(p, q, r)$ and ϕ are the same.