MATH 4LT3/6LT3 Assignment #2

Due: Friday, October 6, 11:59pm

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

1. In the text, the Kuratowski definition of an ordered pair is given along with a proof that it satisfies the ordered pair properties (OP1) and (OP2).

   (a) Show that the following construction also satisfies these properties, where $0 = \emptyset$, and $1 = \{\emptyset\}$:
   $$(x, y) = \{\{x, 0\}, \{y, 1\}\}.$$

   (b) Determine if the following construction satisfies the ordered pair properties:
   $$(x, y) = \{x, \{x, y\}\}.$$

2. Recall the definition of a cardinal assignment from the lectures. Given such an assignment (weak or strong), show that if κ, λ, and μ are cardinals, then $(\kappa \cdot \lambda)^\mu =_c \kappa^\mu \cdot \lambda^\mu$ and $(\kappa^\lambda)^\mu =_c \kappa^{\lambda\mu}$.

3. With $\mathfrak{c}$ the cardinality of the continuum (technically, $|\mathcal{P}(\mathbb{N})|$), show that $\mathfrak{c}^{\mathfrak{c}} =_c 2^{\mathfrak{c}}$. You might consider using the results from the previous problem, and also first establishing that $\aleph_0 \cdot \mathfrak{c} =_c \mathfrak{c}$.

4. For $k \in \mathbb{N}$, let the function $f_k : \mathbb{N} \to \mathbb{N}$ be defined by
   $$f_k(n) = k^{(k^{(k^{\cdot^{\cdot^{\cdot^{k)^{\cdot^{\cdot^{\cdot)}}}}}}}})},$$
   where k appears n-times. Show that there exists a function f with domain $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ such that $f(k, n) = f_k(n)$. You should consider using the recursion with parameters theorem for this. Next, show that the function
   $$e(n) = n^{(n^{(n^{\cdot^{\cdot^{\cdot^{n)^{\cdot^{\cdot^{\cdot)}}}}}}}})},$$
   where n appears n-times, is a member of the set $(\mathbb{N} \to \mathbb{N})$.

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5. Now that we have constructed the natural numbers, i.e., the structured set $(\mathbb{N}, +, \times, \leq, 0, 1)$, show that the integers $(\mathbb{Z}, +, \times, \leq, 0, 1)$ can be faithfully represented as a structured set within our set theoretic universe $\mathcal{W}$. You will need to describe a construction of the integers, along with the operations of $+$, $\times$, the relation $\leq$, and the elements 0 and 1 from the natural numbers, that can be carried out using, indirectly, the Axioms.

6. Let $(P, \leq)$ and $(Q, \preceq)$ be linearly ordered sets.

   (a) Define their sum to be the order over the disjoint union of P and Q such that elements of P are less than all of the elements of Q, and elements within P or Q are ordered according to $\leq$ or $\preceq$ respectively.

   Show that the sum of $(P, \leq)$ and $(Q, \preceq)$ is a linear order. If they are both well orders, is their sum?

   (b) Define the product of these linearly ordered sets to be the order $\sqsubseteq$ on the set $P \times Q$ such that $(p, q) \sqsubseteq (p', q')$ if and only if $(q \prec q')$ or $(q = q'$ and $p \leq p')$.

   Show that the product of $(P, \leq)$ and $(Q, \preceq)$ is a linear order. If they are both well orders, is their product?