MATH 4LT3/6LT3 Assignment #3
Due: Friday, October 27, 11:59pm
Upload your solutions to the Avenue to Learn course website.
Detailed instructions will be provided on the course website.

1. Determine which of the following binary relations are well orderings. Justify your answers.

   (a) The relation $\preceq$ on $\mathbb{N}$, where $n \preceq m$ if and only if (n is even and m is odd) or (n and m are both even or both odd, and $n \leq m$).

   (b) Let Σ be the usual set of 26 letters $\{a, b, c, \ldots, z\}$ and let S be the set of finite strings over Σ (so $S = \Sigma^*$). Let $\leq$ on S be the usual alphabetical ordering. So,

   $$\text{aaaa} < \text{add} < \text{addition} < \text{algebra} < \text{b} < \text{set} < \text{theory},$$

   for example.

   should be < not "less than or equal to".

   (c) For $n \in \mathbb{N}^+ = \{n \in \mathbb{N} \mid n > 0\}$, let $n^\#$ be the number of distinct prime factors of n. The relation $\sqsubseteq$ on $\mathbb{N}^+$ defined by $n \sqsubseteq m$ if and only if $n^\# \leq m^\#$ or ($n^\# = m^\#$ and $n \leq m$).

2. In Assignment #2, Question 6, the sum and product of linear orders was defined. It was shown that if $(U, \leq)$ and $(V, \preceq)$ are well orders then so are their sums and products. Denote their sum and product by $(U, \leq) + (V, \preceq)$ and $(U, \leq) \times (V, \preceq)$, respectively.

   Let $(U, \leq)$, $(V, \preceq)$, and $(W, \sqsubseteq)$ be well orders.

   a "picture proof" is okay for this question.

   (a) Show that

   $$(U, \leq) \times ((V, \preceq) + (W, \sqsubseteq)) =_o ((U, \leq) \times (V, \preceq)) + ((U, \leq) \times (W, \sqsubseteq)).$$

   (b) Does the identity

   $$((U, \leq) + (V, \preceq)) \times (W, \sqsubseteq) =_o ((U, \leq) \times (W, \sqsubseteq)) + ((V, \preceq) \times (W, \sqsubseteq))$$

   also hold?

3. Consider the usual ordering $\leq$ on $\mathbb{R}$. Show that if X is a nonempty subset of $\mathbb{R}$ such that the restriction of $\leq$ to X is a well ordering, then X must be finite or countably infinite.

4. A quasi-order on a set X is a binary relation $\preceq$ on X that satisfies: $x \preceq x$ for all $x \in X$, and if $x, y, z \in X$ with $x \preceq y$ and $y \preceq z$ then $x \preceq z$. So, any partial order on X is a quasi-order on X.

   (a) Let V be a vector space over some field $\mathbb{F}$ and define $\preceq$ on $\mathcal{P}(V)$ by $A \preceq B$ if $\text{Span}(A) \subseteq \text{Span}(B)$. Show that $\preceq$ is a quasi-order on $\mathcal{P}(V)$. Is it a partial order in general?

   (b) Let $\preceq$ be a quasi-order on the set X an define $\sim$ to be the following binary relation on X: $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$. Show that $\sim$ is an equivalence relation on X.

   (c) For X, $\preceq$ and $\sim$ as in the previous part, and $a \in X$, let $[a/\sim]$ denote the equivalence class of $\sim$ that contains a, and let $[X/\sim]$ be the set $\{[a/\sim] \mid a \in X\}$.

   Define the binary relation $\leq$ on $[X/\sim]$ by $[a/\sim] \leq [b/\sim]$ if and only if $a \preceq b$. Show that $\leq$ is a well defined relation on $[X/\sim]$ and that it is a partial order on $[X/\sim]$.

5. Continuing with the previous problem, let $\preceq$ be a quasi-order on the set X. Suppose that $\preceq$ also satisfies these two conditions:

   - For all $x, y \in X$, either $x \preceq y$ or $y \preceq x$.
   - For all subsets A of X, there is some $a \in A$ such that $a \preceq b$ for all $b \in A$.

   (a) Show that under these additional assumptions, that the relation $\leq$ on $[X/\sim]$ is a well ordering.

   (b) Show that the relation $\leq_o$, restricted to the set $WO(A)$ is a quasi-order that satisfies these two additional conditions (you may make use of results from Chapter 7 for some of this). For A a set, $WO(A)$ is the set of well orderings of subsets of A. So

$$WO(A) = \{(U, \sqsubseteq) \mid U \subseteq A \text{ and } \sqsubseteq \text{ is a well ordering of } U\}.$$