Sketching curves

Things to work out (or estimate) when sketching a graph:

- zeroes;
- symmetry;
- inc/dec; min/max;
- concavity;
- behaviour at $\pm\infty$;
- limits at points of undefinedness / discontinuity.

Examples:

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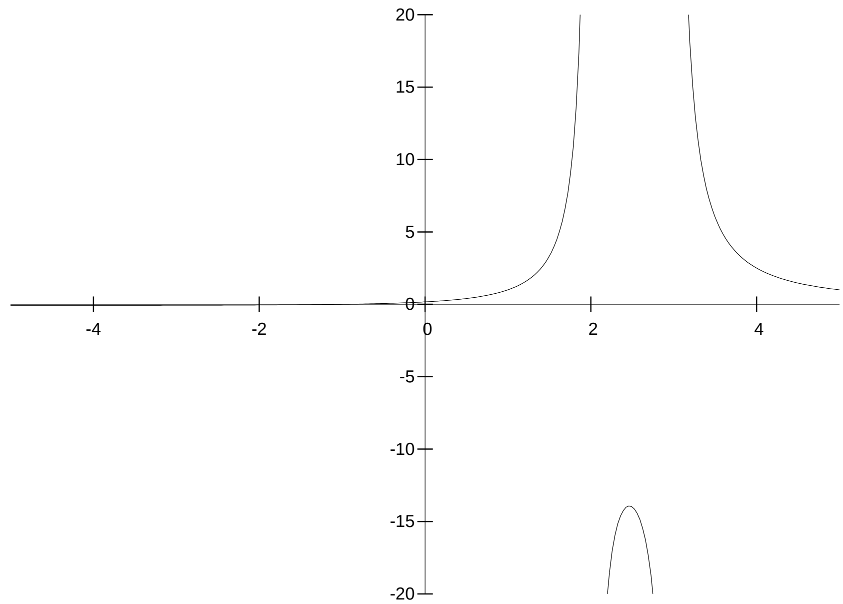
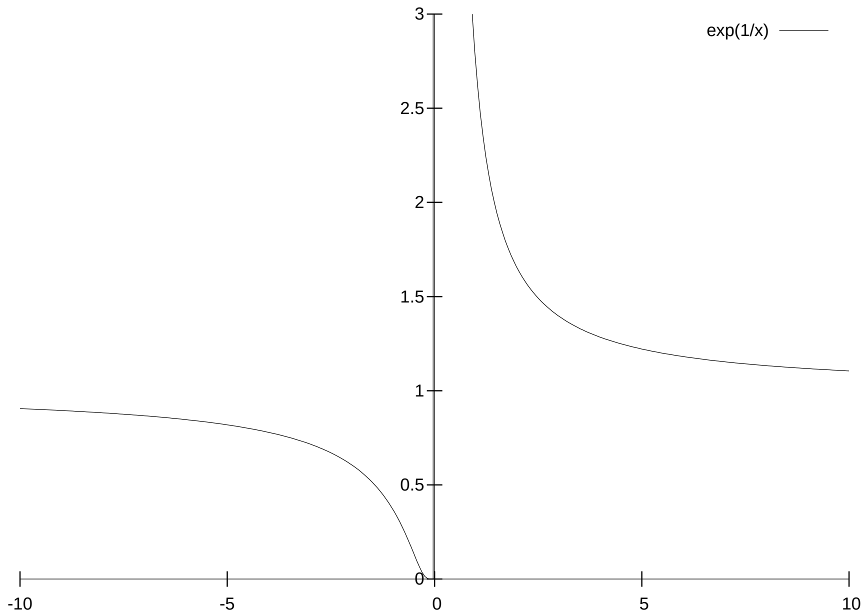
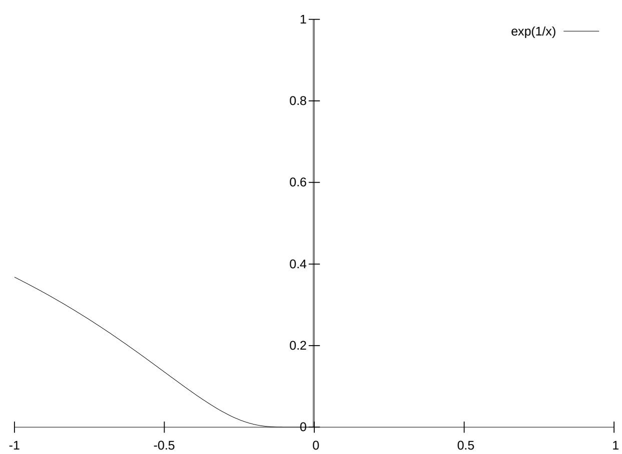
$$f(x) = \frac{x^2 - 1}{(x-1)(x-2)(x-3)}$$



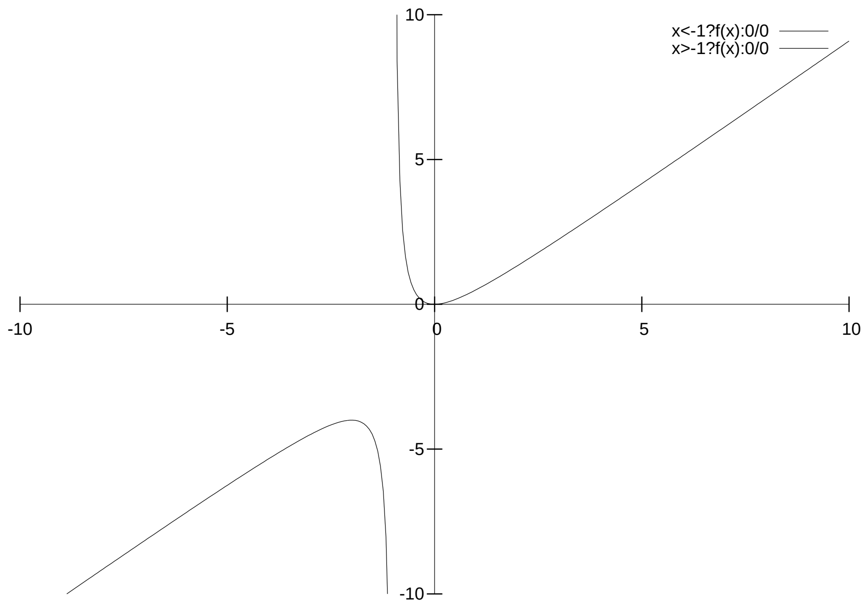
- $$f(x) = e^{\frac{1}{x}}$$





$$\lim_{x \to 0} \left(\frac{d}{dx} e^{\frac{1}{x}} \right) = 0$$

- $$\frac{x^2}{1+x}$$

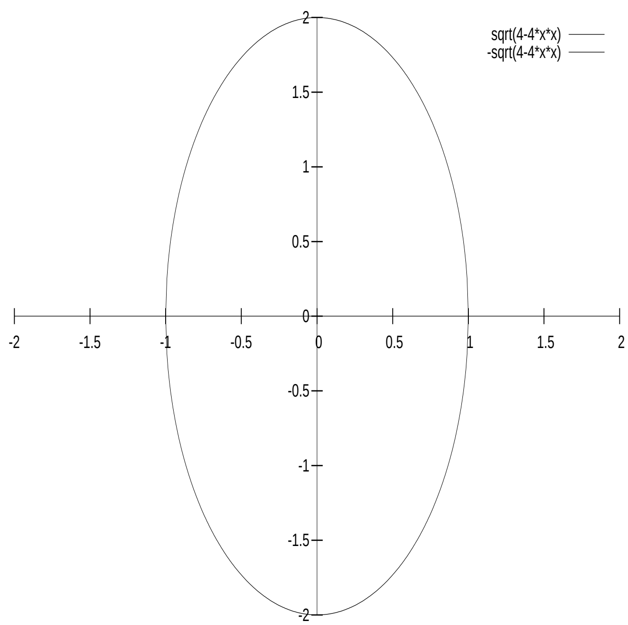


Optimisation

Example: You want to construct a box with a square base and an open top. The volume should be $1m^3$, and you want to minimise the material used. What should be the dimensions of the box?

Example: What are the points on the ellipse furthest away from $(1, 0)$?

$$4x^2 + y^2 = 4?$$



$$\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$$

Antiderivatives

Given $f'(x)$, what could $f(x)$ be?

Example: A particle is accelerated along a line with acceleration $a(t) = t$. What is its velocity $v(t)$ at time t?

One possible answer: $v(t) = \frac{t^2}{2}$ (particle starts from rest).

Another possible answer: $v(t) = \frac{t^2}{2} - 37$ (particle starts with velocity -37).

Definition: f is an <u>antiderivative</u> of g if $f' = g$ (i.e. $f'(x) = g(x)$ for all x).

Uniqueness: "Antiderivatives are unique up to addition of a constant".

If f_1 and f_2 are antiderivatives of g, then for some constant c, $f_2(x) = f_1(x) + c$.

Proof:

$$(f_2(x) - f_1(x))' = f_2'(x) - f_1'(x) = g(x) - g(x) = 0$$

so (by MVT) $f_2(x) - f_1(x)$ is constant, say $f_2(x) - f_1(x) = c$ for all x.

So knowing the value of an antiderivative at a single point determines the antiderivative uniquely.

Example: If $f(x)$ is an antiderivative of $\sin(x)$, and $f(\pi) = 0$, what is $f(x)$?

Solution: We know that $-\cos(x)$ is an antiderivative of $\sin(x)$, so

$$f(x) = -\cos(x) + c$$

for some real number c.

Now $0 = f(\pi) = -\cos(\pi) + c = -(-1) + c = 1 + c$, so $c = -1$.

So $f(x) = -\cos(x) - 1$.

Example: A lunar astronaut throws a ball; it experiences vertical acceleration $y''(t) = -1.62$, but no air resistence. If the initial vertical velocity is $y'(0) = 1$ and the initial horizontal velocity is $x'(0) = 2$, and its initial position is $(x(0), y(0)) = (0, 1)$, and the ground is at $y = 0$, what is the ball's trajectory? How high does it go, and when and where does it hit the ground?

Solution: $y''(t) = -1.62$, so $y'(t) = -1.62t + c_1$. But $y'(0) = 1$, so $c_1 = 1$ and $y'(t) = 1 - 1.62t$. So $y(t) = t - 0.81t^2 + c_2$, and since $y(0) = 1$ we have $y(t) = 1 + t - 0.81t^2$.

Meanwhile, $x''(t) = 0$, so $x'(t) = x'(0) = 2$, so $x(t) = 2t + c_3$, and $x(0) = 0$ so $x(t) = 2t$. So

$$(x(t), y(t)) = \left(2t, 1 + t - 0.81t^2\right)$$

Max height: $y'(t) = 1 - 1.62t = 0 \leftrightarrow t = \frac{1}{1.62} = 0.62$, so at $t = 0.62$ the ball achieves its maximum height $y(0.62) = 1 + 0.62 - 0.81(0.62)^2 = 1.31$.

Hits ground: $y(t) = 0 \leftrightarrow t = \frac{-1 \pm \sqrt{1^2 + 4(0.81)}}{2(-0.81)} \leftrightarrow t = -0.65 \text{ or } t = 1.89$; negative time is irrelevant to the problem; the ball hits the ground at $t = 1.89$, $x = 2(1.89) = 3.78$.

Plotting trajectory: $t = \frac{x}{2}$, so $y(x) = 1 + \frac{x}{2} - 0.81\left(\frac{x}{2}\right)^2$.