Associated Primes of Powers of the Alexander Dual of Path Ideals of Trees

by
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for the degree of Master of Science

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Abstract

Let G be a finite simple graph on the vertex set V_G with the edge set E_G. Using the paths of G of length t, we can generate a square-free monomial ideal $I_t(G)$. Let $I_t(G)^\vee$ denote the square-free Alexander dual of $I_t(G)$. In this project, we study properties of the sets of associated primes $\mathrm{Ass}(R/(I_t(G)^\vee)^s)$, as s increases. We will establish a lower bound on the index of stability N for $\mathrm{Ass}(R/(I_2(G)^\vee)^s)$, when G is a star. As well, we will show how our result can be applied to answer a question raised by Francisco, Hà, and Van Tuyl.

Acknowledgements

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Finally, my most heartfelt thank you to my supervisor Dr. Adam Van Tuyl who has not only become one of my most cherished role models, but also a friend. Thank you for all of your patience, insight, and good humour over the past two years. You have inspired me in many ways and it has been nothing but a pleasure getting to know you. Thank you once again for taking a chance on me and providing me with the opportunity to work under your supervision.

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CHAPTER 1

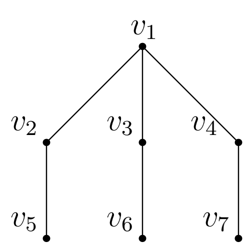
Introduction

The focus of this project is on the algebraic side of the correspondence between commutative algebra and graph theory. We are interested in studying the associated primes of the Alexander dual of path ideals of graph trees.

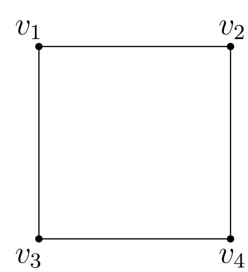
In this chapter we will introduce some terminology from graph theory and commutative algebra and give examples. Detailed definitions can be found in Chapters 2 and 3. Motivation for our problem can be found in Chapter 4, and new results in Chapter 5. Here we will summarize the content of the chapters to follow.

We define a *finite simple graph* G on the vertex set V_G with the edge set E_G as a graph containing no loops or multiple edges. We say that G is *connected* if there exists a path between every pair of vertices $u, v \in V_G$. A graph H is an *induced subgraph* of a graph G if $V_H \subseteq V_G$ and H contains all edges $uv \in E_G$ with $u, v \in V_H$. In the preceding definition, V_H denotes the vertex set of H and V_G and E_G denote the vertex set and edge set of G respectively. If G is a graph on the vertex set $V_G = \{v_1, \ldots, v_n\}$ and the edge set of G is of the form $E_G = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, then we say that G is a *cycle*. A *tree* is a connected graph G that contains no cycles as induced subgraphs.

Example 1.1. Below we have examples of a tree and a cycle of length 4. Both of these graphs are connected.



A tree.



A cycle of length 4.

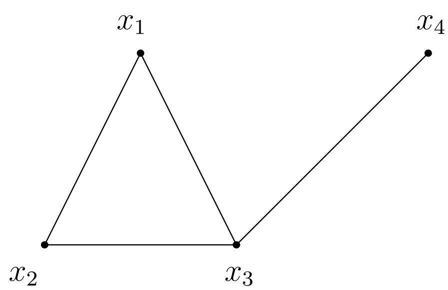
We can make a correspondence between graph theory and commutative algebra by associating graphs with monomial ideals. Specifically, given a finite simple graph G on n vertices, we can generate monomial ideals in the polynomial ring $R = k[x_1, \ldots, x_n]$ over an arbitrary field k using the edges of G.

A *monomial* in R is of the form $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where each $\alpha_i \in \mathbb{N}$. A monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is *square-free* if $\alpha_i = 0$ or $\alpha_i = 1$ for each $i = 1, \ldots, n$. We define a (*square-free*) *monomial ideal* as any ideal generated by a set of (square-free) monomials. The monomial ideal associated to the set of edges of a finite simple graph G, called the *edge ideal*, was first introduced by Villarreal in [**25**].

Definition 1.2. Let G be a finite simple graph on the vertex set $V_G = \{x_1, \ldots, x_n\}$ with the edge set E_G. We define the *edge ideal* corresponding to G by

$$I(G) = (x_i x_j \mid x_i x_j \in E_G).$$

Example 1.3. Let G be the graph shown below. We label the vertices of G with the variables of the polynomial ring $k[x_1, x_2, x_3, x_4]$.



G

The edge ideal corresponding to G is given by

$$I(G) = (x_1x_2, x_1x_3, x_2x_3, x_3x_4),$$

where each $x_i x_j \in E_G$.

By recognizing that an edge can also be viewed as a path of length one, we are able to extend the idea of an edge ideal to a *path ideal*; this concept was first introduced by Conca and De Negri in [**8**].

Definition 1.4. Let G be a finite simple graph on the vertex set $V_G = \{x_1, \ldots, x_n\}$ with the edge set E_G. We define the *path ideal* corresponding to G by

$$I_t(G) = (x_{i_1} \cdots x_{i_{t+1}} \mid \{x_{i_1}, \ldots, x_{i_{t+1}}\} \text{ is a path of } G \text{ of length } t).$$

So when $t = 1$, the path ideal of G is exactly the edge ideal corresponding to G. Path ideals were further studied by Alilooee and Faridi [**1**], Bouchat, Hà, and O'Keefe [**5**], Brumatti and da Silva [**7**], and He and Van Tuyl [**19**].

Every square-free monomial ideal I in R has a dual square-free monomial ideal $I^\vee$ called the *Alexander dual* of I. In the case that I_t is a path ideal, the Alexander dual is given by

$$I_t^\vee = \bigcap_{\{x_{i_1}, \ldots, x_{i_{t+1}}\} \text{ a path of length } t} (x_{i_1} \ldots x_{i_{t+1}}).$$

This project focuses on the associated primes of $I_t^\vee$. We say that an ideal P in a commutative ring R is *prime* if whenever $fg \in P$, either $f \in P$ or $g \in P$. We say that P is an *associated prime* of an ideal I if $P = I : (f)$ for some $f \in R$, where $I : (f) = (g \in R \mid fg \in I)$ is the *ideal quotient* of I and (f). We denote the *set of associated primes* of I by $\mathrm{Ass}(R/I)$.

In [**6**], Brodmann proved that for any ideal I in a commutative Noetherian ring R, there exists an integer N such that

$$\bigcup_{s=1}^{\infty} \mathrm{Ass}(R/I^s) = \bigcup_{s=1}^{N} \mathrm{Ass}(R/I^s).$$

We call the smallest such N the *index of stability*. Although Hoa [**20**] was able to find a bound on the index of stability, computing N in general is very difficult. Instead we strive to understand when the set of associated primes of an ideal satisfies the *persistence property*. We say that an ideal $I \subset R$ satisfies the *persistence property* for associated primes if

$$\mathrm{Ass}(R/I^s) \subseteq \mathrm{Ass}(R/I^{s+1})$$

for all $s \geq 1$. Martinez-Bernal, Morey, and Villarreal [**21**] studied this problem for the edge ideal $I(G)$ and Francisco, Hà, and Van Tuyl have some partial results for $I(G)^\vee$ in [**12**], [**13**], and [**14**].

We are interested in the behaviour of the associated primes of $I_t^\vee$ for values of $t \geq 2$. In particular, we aim to generalize properties of the associated primes of $(I_t^\vee)^s$ as s varies when G is a tree. To simplify notation, let $J_t = I_t^\vee$. When $t = 1$, the Alexander dual J_1, is the *cover ideal* of a graph G which implies that the generators of J_1 are associated to *vertex covers* of G. A *vertex cover* of a graph G is a subset $W \subseteq V$ such that for each $e \in E$, $e \cap W \neq \emptyset$. We will discuss the results that have been found in this case by Francisco, Hà, and Van Tuyl [**12**], [**13**] in Chapter 4.

In general, describing the generators of J_t for values of $t \geq 2$ corresponding to an arbitrary graph G is difficult. In this project, we focused on the particular case in which we treat a *star* as a tree which is not rooted at any vertex. A *star* is a special case of a *complete bipartite graph*. A *complete bipartite graph* $K_{m,n}$, is a bipartite graph with vertex partition $V = V_1 \cup V_2$ where $|V_1| = m$ and $|V_2| = n$, such that every $x_i \in V_1$ and $x_j \in V_2$ are adjacent, i.e., are connected by an edge. When $m = 1$, we say that $K_{1,n}$ is a *star* and we call the single vertex in V_1 the star's *centre*. We find an expression for the generators of J_2 in the following lemma:

Lemma 1.5. *Let $K_{1,n}$ be a star on vertex set $V = \{z, x_1, \ldots, x_n\}$ with centre z. Then*

$$J_2 = (z) + (x_1 \cdots \hat{x}_i \cdots x_n \mid i = 1, 2, \ldots, n)$$

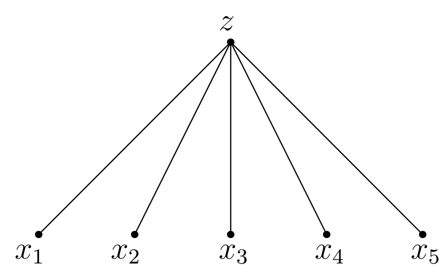
where $\hat{x}_i$ denotes that the vertex x_i is removed from the product.

We will demonstrate this idea with an example.

Example 1.6. Let $K_{1,5}$ be a star on the vertex set $V = \{z, x_1, x_2, x_3, x_4, x_5\}$ with centre z as shown below. When we regard $K_{1,5}$ as a tree which is not rooted, we obtain the

2-path ideal

$$I_2 = (x_1zx_2, x_1zx_3, x_1zx_4, x_1zx_5, x_2zx_3, x_2zx_4, x_2zx_5, x_3zx_4, x_3zx_5, x_4zx_5).$$



$K_{1,5}$

The corresponding Alexander dual of I_2 is given by

$$J_2 = (z, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4x_5, x_1x_3x_4x_5, x_2x_3x_4x_5).$$

When $s = 1$, $\mathrm{Ass}(R/J_2)$ is comprised of all prime ideals of the form (x_i, z, x_j) where $i \neq j$. Then as we increase the value of s, we will find that the maximal ideal $(z, x_1, x_2, x_3, x_4, x_5)$, appears in $\mathrm{Ass}(R/J_2^4)$. Moreover, $(z, x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}(R/J_2^s)$ for all $s \geq 4$.

The observation that $(z, x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}(R/J_2^s)$ for all $s \geq 4$ in Example 1.6 led us to the following theorem:

Theorem 1.7. *Let $K_{1,n}$ be a star on the vertex set $V = \{z, x_1, \ldots, x_n\}$ with centre z and corresponding Alexander dual J_2. Then*

(1) $(z, x_1, \ldots, x_n) \in \mathrm{Ass}(R/J_2^s)$ *for all* $s \geq n - 1$.
(2) $(z, x_1, \ldots, x_n) \notin \mathrm{Ass}(R/J_2^s)$ *for all* $s < n - 1$.

The proofs of Lemma 1.5 and Theorem 1.7 can be found in Chapter 5. Using Theorem 1.7, we can describe a lower bound on the index of stability N when our given graph is a star.

Corollary 1.8. *Let $K_{1,n}$ be a star on the vertex set $\{z, x_1, \ldots, x_n\}$ with centre z and Alexander dual J_2. Then*

$$N \geq n - 1$$

where N denotes the index of stability of $\mathrm{Ass}(R/J_2^s)$.

Corollary 1.8 tells us that if $K_{1,n}$ is a star on $n+1$ vertices where $n \gg 2$, then a lower bound on N will be quite large.

The final chapter of this project will introduce a number of open questions. The conjectures made in Chapter 6 are based on observations made while conducting computer experiments. Working on these problems in the future will hopefully help us to generalize our results from Chapter 5 to a larger family of graphs.

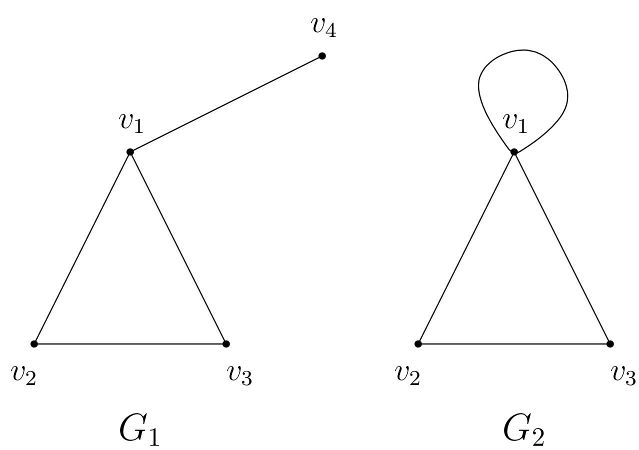
CHAPTER 2

Some Graph Theory

We begin by introducing some basic ideas from graph theory. We are primarily concerned with properties of *simple graphs*. In this section, we will define a simple graph and some of its properties. There are several families of simple graphs, a few of which will be introduced in sections 2.1, 2.2, and 2.3. The following definitions can be found in [**3**], [**10**], [**13**] and [**26**].

Definition 2.1 (Simple Graph)**.** A *simple graph* G with n *vertices* and m *edges* has a *vertex set* $V_G = \{v_1, \ldots, v_n\}$ and an *edge set* $E_G = \{e_1, \ldots, e_m\}$, where each edge is an unordered pair of distinct vertices. We denote the edge $e = \{u, v\}$ by uv and if $uv \in E_G$, then we say that u and v are *adjacent*. We also say that a graph G is a simple graph if G has no loops or multiple edges.

Example 2.2. In the figure below, G_1 is a simple graph whereas G_2 is not since there is an edge from v_1 to itself (a loop).



We will often refer to a simple graph G as a *finite simple graph*. The word *finite* refers to the fact that $|V_G| < \infty$. There are many families of finite simple graphs which will be discussed throughout the rest of this chapter. However, in Chapters 4 and 5 we will focus mainly on the properties of *trees* (Definition 2.19).

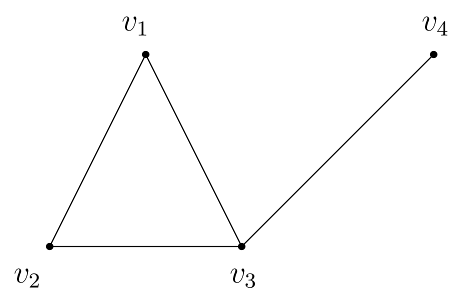
From now on, unless it is necessary to specify, we will write $G = (V_G, E_G)$ to denote a simple graph G on a finite vertex set V_G with an arbitrary edge set E_G.

Definition 2.3 (Path)**.** Let u and v be (not necessarily distinct) vertices of a graph G. A *u-v path* is an alternating sequence of vertices and edges beginning at u (called the *start vertex*) and ending at v (called the *end vertex*) such that no vertex is repeated. The *length* of a u-v path is the number of edges in the sequence. We represent a u-v path of G on the vertices $u = v_1, v_2, \ldots, v_k = v \in V_G$ by $v_1v_2 \cdots v_k$ where each $v_iv_{i+1} \in E_G$ for $1 \leq i \leq k-1$.

Definition 2.4 (Connected Graph)**.** Let $G = (V_G, E_G)$. We say that G is *connected* if there exists a u-v path for every pair of vertices $u, v \in V_G$.

We illustrate the definitions of a path and a connected graph in the following example:

Example 2.5. Suppose that G is the connected graph shown below on the vertex set $V_G = \{v_1, v_2, v_3, v_4\}$ with the edge set $E_G = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4\}$. It is easy to see that for any pair of vertices $v_i, v_j \in V_G$, we can find a v_i-v_j path of G. For example a v_1-v_4 path of length 2 is given by $v_1v_3v_4$.



G

Every graph $G = (V_G, E_G)$ is either *directed* or undirected. If G is undirected, we can assign an *orientation* to each $e \in E_G$ and obtain a directed graph. We define a directed graph and an orientation of an undirected graph below and give an example.

Definition 2.6 (Directed Graph)**.** A *directed graph* (or *digraph*) is a graph G on a vertex set V_G with an edge set E_G such that there exist two maps $\text{init} : E_G \to V_G$ and $\text{ter} : E_G \to V_G$ assigning every edge e to an *initial vertex* $\text{init}(e)$ and a *terminal vertex* $\text{ter}(e)$. The edge e is said to be *directed from* $\text{init}(e)$ *to* $\text{ter}(e)$.

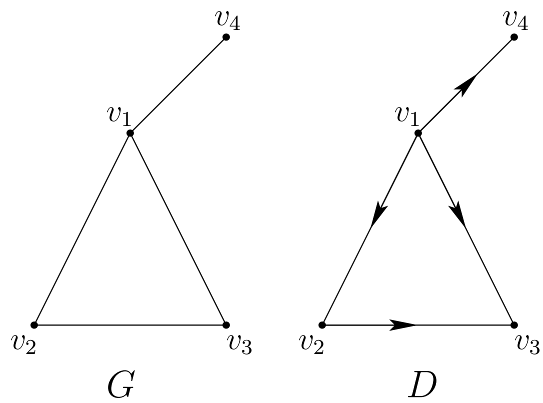
Definition 2.7 (Orientation)**.** A directed graph D is an *orientation* of an (undirected) graph G if $V_D = V_G$ and $E_D = E_G$, and if $\{\text{init}(e), \text{ter}(e)\} = \{u, v\}$ for every edge $e = uv$. An *oriented graph* arises from an undirected graph G by directing every $e \in E_G$ from one of its ends to the other, i.e., if $e = uv$ then we will direct e from u to v or from v to u for every $e \in E_G$. If e is directed from u to v then we denote the directed edge e by (u, v).

Example 2.8. Let $G = (V_G, E_G)$ be the undirected graph as shown below. Then $E_G = \{e_1 = v_1v_2, e_2 = v_1v_3, e_3 = v_1v_4, e_4 = v_2v_3\}$. We will assign an orientation to every $e_i \in E_G$ to obtain the directed graph D.

First we let $V_D = V_G$ and $E_D = E_G$. Now we assign the following orientation to the edges

- $\{\text{init}(e_1), \text{ter}(e_1)\} = \{v_1, v_2\}$
- $\{\text{init}(e_2), \text{ter}(e_2)\} = \{v_1, v_3\}$
- $\{\text{init}(e_3), \text{ter}(e_3)\} = \{v_1, v_4\}$
- $\{\text{init}(e_4), \text{ter}(e_4)\} = \{v_2, v_3\}$.

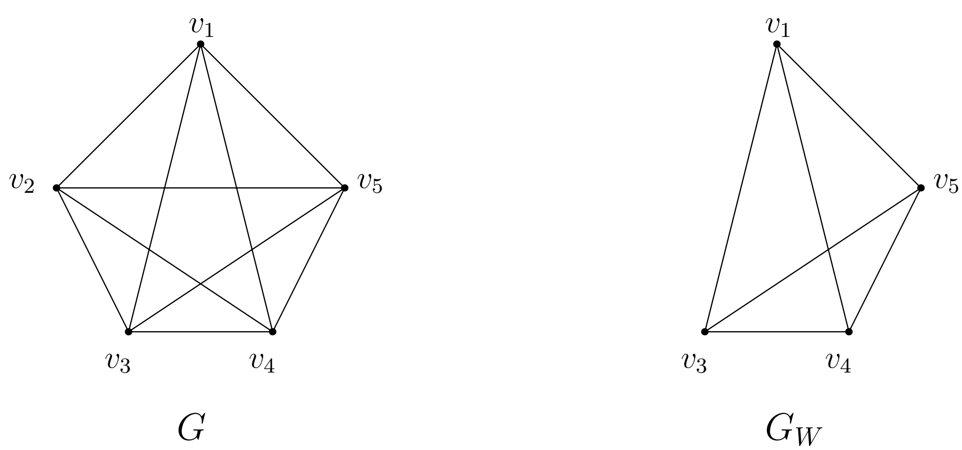
Under this orientation, D is a directed graph. The arrows in D correspond to the orientation of each edge and we denote the edge set of D by $E_D = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3)\}$.



There exist smaller graphs within a given graph G. We call these graphs *subgraphs* of G. Later in this chapter we will see how to determine various properties of the graph G by studying its subgraphs. Below, we define subgraphs as well as give an example.

Definition 2.9 ((Induced) Subgraph)**.** Let $G = (V_G, E_G)$. A *subgraph* of G is a graph H such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$. We write $H \subseteq G$. A subgraph H is an *induced subgraph* of G if H contains all edges $uv \in E_G$ with $u, v \in V_H$.

Given a graph $G = (V_G, E_G)$, we often consider the induced subgraph on a subset $W \subseteq V_G$. We denote the induced subgraph on W by G_W. We demonstrate this concept in the following example:



Example 2.10. Let G be the graph shown above on the left on the vertex set $V_G = \{v_1, v_2, v_3, v_4, v_5\}$ with the edge set

$$E_G = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5, v_4v_5\}.$$

Consider the subset $W = \{v_1, v_3, v_4, v_5\} \subset V_G$. The induced subgraph G_W is given above on the right with the edge set

$$E_{G_W} = \{v_1v_3, v_1v_4, v_1v_5, v_3v_4, v_3v_5, v_4v_5\}.$$

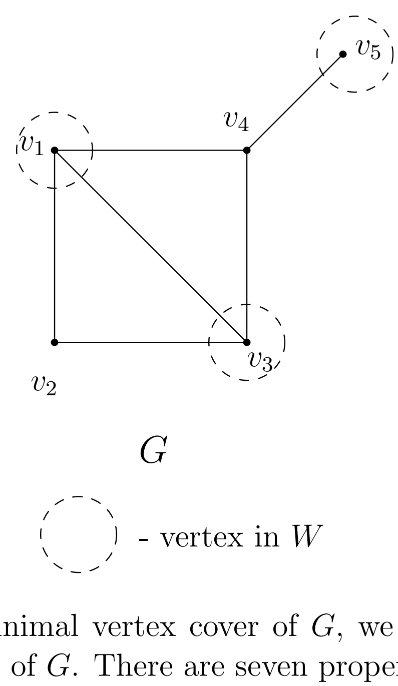
We conclude this section by defining a (*minimal*) *vertex cover* of a graph G and giving an example. We will be using Definition 2.11 several times throughout the rest of this project.

Definition 2.11 ((Minimal) Vertex Cover)**.** Let G be a finite simple graph on the vertex set V_G with the edge set E_G. A subset $W \subseteq V_G$ is a *vertex cover* of G if for every $e \in E_G$, $e \cap W \neq \emptyset$. We say that W is a *minimal vertex cover* of G if no proper subset of W is a vertex cover of G.

Example 2.12. Let G be the graph on the vertex set $V_G = \{v_1, v_2, v_3, v_4, v_5\}$ with the edge set $E_G = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_3v_4, v_4v_5\}$ shown below. We claim that a minimal vertex cover of G is given by $W = \{v_1, v_3, v_5\}$. To see this, we first need to check that W is in fact a vertex cover of G. To simplify notation, let $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_1, v_4\}$, $e_4 = \{v_2, v_3\}$, $e_5 = \{v_3, v_4\}$, and $e_6 = \{v_4, v_5\}$. We have that

$$W \cap e_1 = \{v_1\},\ W \cap e_2 = \{v_1, v_3\},\ W \cap e_3 = \{v_1\},$$
$$W \cap e_4 = \{v_3\},\ W \cap e_5 = \{v_3\},\ \text{and } W \cap e_6 = \{v_5\}.$$

Since $W \cap e_i \neq \emptyset$ for each $i = 1, \ldots, 6$, W is a vertex cover of G.



To check that W is a minimal vertex cover of G, we need to check that no proper subset of W is a vertex cover of G. There are seven proper subsets of W, namely $\emptyset$, $\{v_j\}$

for $j = 1, 3, 5$, $\{v_1, v_3\}$, $\{v_1, v_5\}$, and $\{v_3, v_5\}$. Clearly, $\emptyset \cap e_i = \emptyset$ for all i and therefore $\emptyset$ is not a vertex cover of G. Then since

$$\{v_1\} \cap e_4 = \{v_3\} \cap e_1 = \{v_5\} \cap e_1 = \emptyset,$$

the sets $\{v_1\}$, $\{v_3\}$, and $\{v_5\}$ do not satisfy the definition of a vertex cover of G. Finally,

$$\{v_1, v_3\} \cap e_6 = \{v_1, v_5\} \cap e_4 = \{v_3, v_5\} \cap e_1 = \emptyset$$

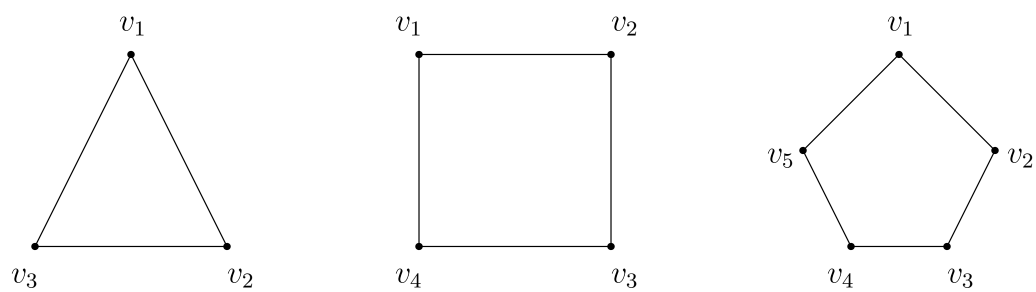
and hence $\{v_1, v_3\}$, $\{v_1, v_5\}$, and $\{v_3, v_5\}$ are not vertex covers of G. Since W is a vertex cover of G and no proper subset of W is a vertex cover of G, we can conclude that W is a minimal vertex cover of G.

1. Cycles, Cliques, and Trees

In this section we introduce three special families of graphs, namely *cycles*, *cliques* and *trees*.

Definition 2.13 (Cycle of Order n)**.** A *cycle of order* n, denoted C_n, is a finite simple graph with the vertex set $V_{C_n} = \{v_1, \ldots, v_n\}$ and the edge set $E_{C_n} = \{v_1v_2, v_2v_3, \ldots, v_nv_1\}$.

Example 2.14. The cycles of orders $3, 4$, and 5, respectively, are shown below.



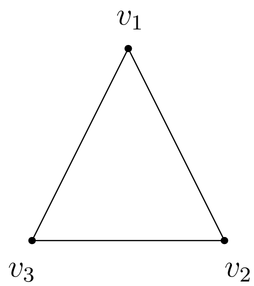
C_3 C_4 C_5

A cycle of order n is a subgraph of a *clique* on n vertices (Definition 2.15). We will see in Example 2.17 how we can obtain a clique by adding edges to a cycle C_n.

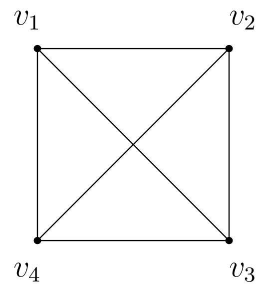
Definition 2.15 (Clique)**.** A simple graph $G = (V_G, E_G)$ is a *clique* (or a *complete graph*) if for every pair of vertices $u, v \in V_G$, u and v are adjacent, i.e., $uv \in E_G$. We denote a clique on n vertices by $\mathcal{K}_n$.

Definition 2.16 (Clique Number)**.** The *clique number* of a graph G is the size of the largest induced clique of G. We denote the clique number of G by $\omega(G)$.

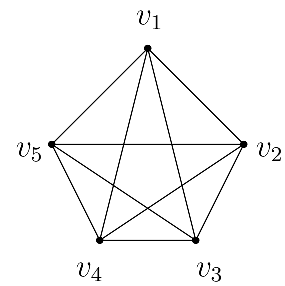
Example 2.17. The cliques of size $3, 4$ and 5, respectively, are shown below. By comparing these figures with the cycles in Example 2.14, we can see that C_3 is in fact a clique of size 3, that is, $\mathcal{K}_3$. If we add the edges v_1v_3 and v_2v_4 to C_4 and the edges $v_1v_3, v_1v_4, v_2v_4, v_2v_5$ and v_3v_5 to C_5, we obtain $\mathcal{K}_4$ and $\mathcal{K}_5$ respectively.



$\mathcal{K}_3$

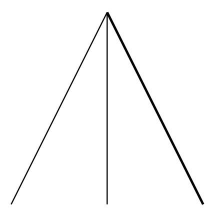


$\mathcal{K}_4$



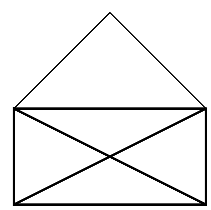
$\mathcal{K}_5$

Example 2.18. We can see that the largest induced clique of G_1 (see figure below) is of size 2 (an edge) so $\omega(G_1) = 2$. Although G_2 has both $\mathcal{K}_3$ and $\mathcal{K}_4$ as induced subgraphs, the size of the largest induced clique of G_2 is 4 and hence $\omega(G_2) = 4$.



G_1

$\omega(G_1) = 2$



G_2

$\omega(G_2) = 4$

Definition 2.19 (Tree)**.** A *tree* Γ is a connected graph which does not contain any cycle as an induced subgraph. We say that Γ is *rooted* if there is a designated vertex v_k such that every v_i-v_j path is naturally oriented away from v_k. If Γ has no such root, then we say that Γ is *not rooted*.

Example 2.20. Let $\Gamma_1 = (V_{\Gamma_1}, E_{\Gamma_1})$ and $\Gamma_2 = (V_{\Gamma_2}, E_{\Gamma_2})$, where

$$V_{\Gamma_1} = V_{\Gamma_2} = \{v_1, v_2, v_3, v_4, v_5, v_6\},$$

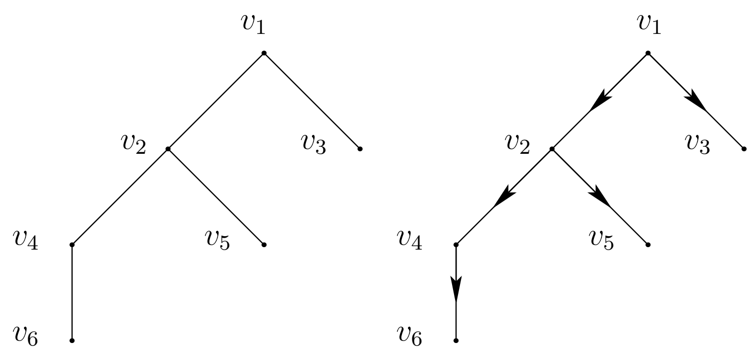
$$E_{\Gamma_1} = \{v_1v_2, v_1v_3, v_2v_4, v_2v_5, v_4v_6\},$$

and

$$E_{\Gamma_2} = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_2, v_5), (v_4, v_6)\}.$$

The graph Γ_1 is an example of a tree which is not rooted while Γ_2 is rooted at the vertex v_1. In the graph Γ_2, we represent the orientation of the edges with arrows while in the edge set E_{Γ_2}, the ordered pair (v_i, v_j) implies that an edge is directed from v_i to v_j.

In Γ_1 we can take the v_2-v_3 path $v_2v_1v_3$ whereas in Γ_2 we cannot since every path of Γ_2 must be oriented away from v_1. There are three paths of length 2 in Γ_2; they are (v_1, v_2, v_4), (v_1, v_2, v_5), and (v_2, v_4, v_6) (note that for a path of length 2 in Γ_2, we have used the triple (v_i, v_j, v_k) to denote a directed path from v_i to v_k, where (v_i, v_j) and (v_j, v_k) are directed edges).



Γ_1 Γ_2

In Example 2.20, we can easily see that the largest induced clique of both Γ_1 and Γ_2 is of size 2 (an edge) so $\omega(\Gamma_1) = \omega(\Gamma_2) = 2$. We conclude this section by showing that $\omega(\Gamma) = 2$ for any tree Γ.

Lemma 2.21. *If Γ is a tree, then $\omega(\Gamma) = 2$.*

Proof. Let Γ be a tree. Suppose that $\omega(\Gamma) \neq 2$. Then $\omega(\Gamma) = k \geq 3$, i.e. Γ contains an induced clique of size $k \geq 3$. Let $W = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V_\Gamma$ be the vertex set of the induced clique of Γ of size k. Then by definition, every pair of vertices v_{i_r}, $v_{i_s} \in W$ is adjacent. In particular, $v_{i_1}v_{i_2}, v_{i_2}v_{i_3}, \ldots, v_{i_{k-1}}v_{i_k}$ are all edges in $E_W \subseteq E_G$. In addition, $v_{i_k}v_{i_1} \in E_W$. So we have a cycle

$$v_{i_1}v_{i_2}\cdots v_{i_k}v_{i_1}$$

in Γ. This contradicts the assumption that Γ is a tree. Hence, Γ contains no cliques of size $k \geq 3$ and thus $\omega(\Gamma) = 2$. □

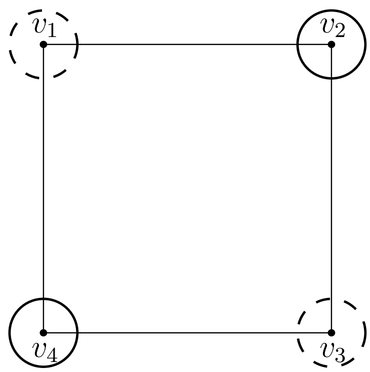
2. Bipartite Graphs

In this section we define and establish several properties of a large family of graphs called *bipartite graphs*.

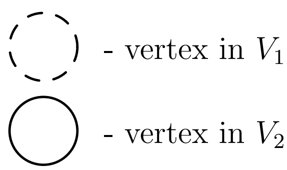
Definition 2.22 (k-partite Graph)**.** Let $k \geq 2$ be an integer. A graph $G = (V_G, E_G)$ is *k-partite* if there exists a partition $V_G = V_1 \cup \cdots \cup V_k$ such that for every $e = uv \in E_G$, $u \in V_i$ and $v \in V_j$ where $i \neq j$. We call $V_1, \ldots, V_k$ *classes* of the partition.

We will consider specifically the case where $k = 2$. We call a 2-partite graph *bipartite*. Here we give some examples of bipartite graphs.

Example 2.23. **(1)** The cycle of order 4, C_4, is a bipartite graph. To see this, consider the partition of the vertex set V_{C_4} into $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$. Then $e \not\subseteq V_i$ for $i = 1, 2$, for any $e \in E_{C_4}$. So C_4 satisfies the definition of a bipartite graph. In the figure below, the vertices in V_1 are circled with a dotted line and the vertices in V_2 are circled with a solid line.

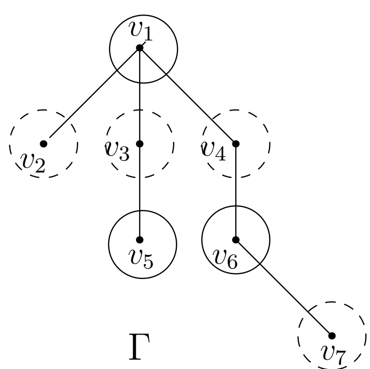


C_4

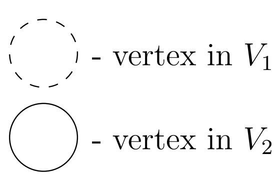


In fact, any C_n of even order is bipartite. We prove this in Corollary 2.27.

(2) Let Γ be the tree shown below on the vertex set $V_\Gamma = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ with the edge set $E_\Gamma = \{v_1v_2, v_1v_3, v_1v_4, v_3v_5, v_4v_6, v_6v_7\}$. We can partition V_Γ into $V_1 = \{v_2, v_3, v_4, v_7\}$ and $V_2 = \{v_1, v_5, v_6\}$ to see that Γ is a bipartite graph.

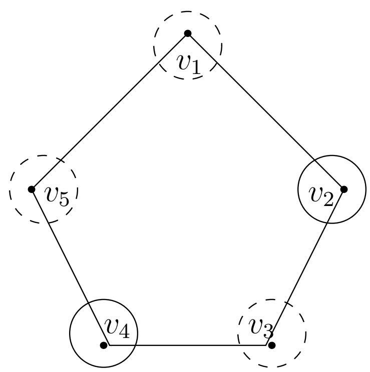


Γ



In Corollary 2.28, we prove that all trees are bipartite.

(3) We will show that C_5 is not bipartite. Let C_5 be the cycle of order 5 shown below on the vertex set $V_{C_5} = \{v_1, v_2, v_3, v_4, v_5\}$ with the edge set $E_{C_5} = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Now suppose that C_5 is bipartite. Then there exists a partition $V_{C_5} = V_1 \cup V_2$ such that $e \not\subseteq V_i$ for $i = 1, 2$ for any $e \in E_{C_5}$. Without loss of generality, let $v_1 \in V_1$. Then since $v_1v_2 \in E_{C_5}$, we must have that $v_2 \in V_2$. Now $v_2v_3 \in E_{C_5}$ so $v_3 \in V_1$. Similarly $v_4 \in V_2$ and $v_5 \in V_1$. But then $v_1, v_5 \in V_1$ and $v_5v_1 \in E_{C_5}$, which contradicts the assumption that C_5 is bipartite. Hence C_5 is not bipartite. We prove in Theorem 2.26 that any graph G containing an odd induced cycle is not bipartite.



C_5

- vertex in V_1

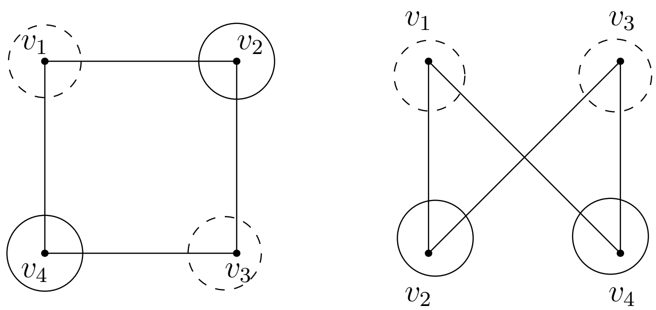
- vertex in V_2

We will now define a *complete bipartite graph* and give an example.

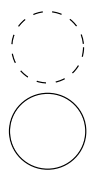
Definition 2.24 (Complete Bipartite Graph)**.** Let $G = (V_G, E_G)$ be a bipartite graph with vertex partition $V = V_1 \cup V_2$. If for every $u \in V_1$ and $v \in V_2$, u and v are adjacent, (that is, $uv \in E_G$), then we say that G is a *complete bipartite graph.* We denote a complete bipartite graph by $K_{m,\ell}$, where $m = |V_1|$ and $\ell = |V_2|$.

Example 2.25. In Example 2.23**(1)**, we saw that C_4 is a bipartite graph. Here, we will show that C_4 is in fact the complete bipartite graph $K_{2,2}$. Let $V_{C_4} = \{v_1, v_2, v_3, v_4\}$ denote the vertex set of C_4 with the edge set $E_{C_4} = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. We want to find a vertex partition $V_{C_4} = V_1 \cup V_2$ such that $|V_1| = |V_2| = 2$ and for every $v_i \in V_1$ and $v_j \in V_2$, $v_iv_j \in E_{C_4}$. Let $v_1 \in V_1$. Then since v_1 is adjacent to both v_2 and v_4, let $v_2, v_4 \in V_2$. Since we require $|V_1| = 2$, let $v_3 \in V_1$. So the sets in our partition are $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$. We need to check that every $v_i \in V_1$ and $v_j \in V_2$ are adjacent. We know that v_1 is adjacent to both v_2 and v_4 in V_2 by the way we chose our partition.

Now $v_2v_3 \in E_{C_4}$ and $v_3v_4 \in E_{C_4}$ so v_3 is adjacent to every vertex in V_2. It follows that $C_4 = K_{2,2}$. The graphs are shown below.



C_4 $K_{2,2}$



- vertex in V_1

- vertex in V_2

Theorem 2.26. *A connected finite simple graph G is bipartite if and only if G contains no odd cycles.*

Proof. Suppose that G is a bipartite graph with the vertex set V_G and the edge set E_G and suppose that G contains an odd induced cycle C_{2k-1} on the vertices $\{v_{i_1}, \ldots, v_{i_{2k-1}}\}$ for some $k \geq 2$. Since G is bipartite, there exists subsets $V_1, V_2 \subseteq V$ such that $V = V_1 \cup V_2$ and $e \not\subseteq V_i$ for $i = 1, 2$ and for any $e \in E_G$. Without loss of generality, suppose that $v_{i_1} \in V_1$. Then since $v_{i_1}v_{i_2} \in E_G$ we must have that $v_{i_2} \in V_2$. Now consider the edge $v_{i_2}v_{i_3}$. Since $v_{i_2} \in V_2$, we have that $v_{i_3} \in V_1$. Continuing in this way we will have that $v_{i_{2\ell-1}} \in V_1$ and $v_{i_{2\ell}} \in V_2$ for all $\ell \geq 1$. Now we consider the edge $v_{i_1}v_{i_{2k-1}}$ that closes the cycle. We know that $v_{i_1} \in V_1$ and since $v_{i_{2k-1}}$ has an odd index, we must have that $v_{i_{2k-1}} \in V_1$. But now we have an edge in V_1 which contradicts the fact that G is bipartite. Hence, G contains no odd cycles.

Now suppose that G contains no odd cycles and fix a vertex $u \in V_G$. We claim that the following partition will make G bipartite:

- $V_1 = \{v \in V_G \mid \text{shortest } u - v \text{ path has even length}\}$
- $V_2 = \{w \in V_G \mid \text{shortest } u - w \text{ path has odd length}\}$.

Note that $u \in V_1$. Now since every path must be of either even or odd length, we have $V_G = V_1 \cup V_2$ and since a path cannot be both even and odd, $V_1 \cap V_2 = \emptyset$. So we need to show that $e \not\subseteq V_i$ for $i = 1, 2$ for any $e \in E_G$. We do this by contradiction. Suppose that $v, w \in V_1$ such that $vw \in E_G$. Let P and Q be the shortest $u - v$ and $u - w$ paths respectively, say

$$P = u_1, u_2, \ldots, u_{2m+1} \qquad\qquad Q = w_1, w_2, \ldots, w_{2n+1}$$

where $u = u_1 = w_1, v = u_{2m+1}$, and $w = w_{2n+1}$. Let w' denote the last common vertex of P and Q. (Such a vertex exists since P and Q always have vertex u in common. We also note that if $v = w$, then $w' = v = w$.) Then the part of P from u to w' is the shortest $u - w'$ path and similarly the part of Q from u to w' is the shortest $u - w'$ path. Since these paths must have the same length, there exists an index i such that $w' = u_i = w_i$. But then

$$C = \underbrace{u_i, u_{i+1} \cdots u_{2m+1}}_{*} \underbrace{w_{2n+1} w_{2n} \cdots w_i}_{**}$$

is a cycle. Now if i is odd, then both $*$ and $**$ have even length and if i is even, both $*$ and $**$ have odd length. So the length of C has parity

$$\begin{aligned} &\text{even} + 1 + \text{even} = \text{odd} \quad \text{or} \\ &\text{odd} + 1 + \text{odd} = \text{odd}. \end{aligned}$$

We have found an odd cycle of G which is a contradiction. So we cannot have $v, w \in V_1$ where $vw \in E_G$. Using the same argument, we can show that $e \not\subseteq V_2$ for any $e \in E_G$. Thus, G is bipartite. □

Corollary 2.27. *Even cycles are bipartite.*

Proof. Since an even cycle will not contain an odd cycle, it follows immediately from Theorem 2.26 that even cycles are bipartite. □

Corollary 2.28. *Trees are bipartite.*

Proof. Let Γ be a tree. By definition, Γ contains no odd cycles. It follows from Theorem 2.26 that Γ is bipartite. □

3. Graph Colourings

In this section we describe a *colouring* of a graph G and establish the *chromatic number* of bipartite graphs.

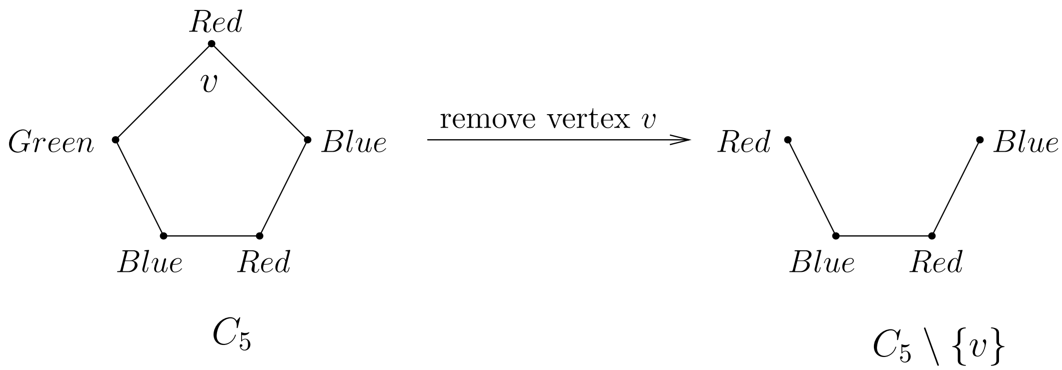
Definition 2.29 (d-colouring)**.** Let $G = (V_G, E_G)$ be a finite simple graph. Then a *d-colouring* is any partition, $V_G = C_1 \cup \cdots \cup C_d$ into d disjoint sets such that for every $e \in E_G$, we have $e \not\subseteq C_i$ for all $i = 1, \ldots, d$. We call the C_i's the *colour classes* of a d-colouring.

Definition 2.30 (Chromatic Number)**.** Let G be a finite simple graph. The *chromatic number* $\chi(G)$, of G is the minimum number of colours needed to colour the vertices of G such that no two adjacent vertices receive the same colour. If $\chi(G) = d$, then we say that G is *d-chromatic*.

Definition 2.31 (Critically d-chromatic)**.** We say that G is *critically d-chromatic* if $\chi(G) = d$ but for every $v \in V_G$, $\chi(G \setminus \{v\}) < d$, where $G \setminus \{v\}$ is the graph G with the vertex v and every edge containing v removed.

Example 2.32. Consider the cycle of order 5 as shown below. We can label the vertices of the cycle using three colours so C_5 is 3-chromatic.

We are curious as to whether or not C_5 is *critically* 3-chromatic so we remove vertex v. Now we can colour $C_5 \setminus \{v\}$ with two colours so $\chi(C_5 \setminus \{v\}) = 2 < 3$. Similarly, if we remove any other vertex u from C_5 we can colour $C_5 \setminus \{u\}$ with only two colours. Hence, C_5 is critically 3-chromatic.



We conclude this section by showing that bipartite graphs, and consequently trees, are 2-chromatic.

Theorem 2.33. *If $G = (V_G, E_G)$ is a bipartite graph, then $\chi(G) = 2$.*

Proof. Since G is bipartite, there exists independent sets V_1, V_2 that partition V_G so we can write $V_G = V_1 \cup V_2$. Now let $C_1 = V_1$ and $C_2 = V_2$. We claim that we need at least two colours to colour the vertices of G such that no two adjacent vertices receive the same colour, i.e., $\chi(G) = 2$. Seeking a contradiction, suppose that $\chi(G) = 1$. Let C_1 represent this colour class and let $u \in C_1$. Now consider $uv \in E_G$. Since G is bipartite, we must have that $v \in V_2$. However, $\chi(G) = 1$ so v must receive the same colour as u, i.e., $v \in C_1 = V_1$. Now $uv \in E_G$ and $\{u, v\} \subseteq V_1$ which contradicts the fact that G is bipartite. Hence, $\chi(G) = 2$. $\square$

Corollary 2.34. *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a tree. Then $\chi(\Gamma) = 2$.*

Proof. It follows from Corollary 2.28 that Γ is bipartite. Then by Theorem 2.33, we can conclude that $\chi(\Gamma) = 2$. $\square$

CHAPTER 3

Commutative Algebra: Primary Decomposition and Monomial Ideals

We move away from graph theory for a moment to introduce the tools we will be using from commutative algebra. Throughout this section let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over an arbitrary field k. We will be using definitions from [**11**], [**15**] and [**18**].

Here we introduce various types of *ideals* $I \subset R$ and define the *associated primes* of an ideal I.

Definition 3.1 (Ideal)**.** Let $I \subset R$. Then I is an *ideal* of R if

(1) $I \neq \emptyset$,
(2) $f - g \in I$ for all $f, g \in I$, and
(3) $hf \in I$ for all $h \in R$ and $f \in I$.

Let $\{f_1, \ldots, f_r\}$ be a finite collection of polynomials in R. If

$$I = \{g_1 f_1 + \cdots + g_r f_r \mid g_i \in R\},$$

then we say that I is *generated by* $f_1, \ldots, f_r$ and we write

$$I = (f_1, \ldots, f_r).$$

Definition 3.2 (Principal Ideal)**.** Let $I \subset R$ be an ideal. If $I = (f)$ for some $f \in R$, then I is a *principal ideal.*

Example 3.3. Let $I = (x, x^3, xy) \subset k[x, y]$. We claim that I is a principal ideal generated by x. We use double inclusion to show that $I = (x)$. Since x divides every generator of I, we have that $I \subseteq (x)$. Then since $x \in I$, it follows that $(x) \subseteq I$ and we have the second inclusion. Hence, $I = (x)$ is a principal ideal.

Definition 3.4 (Maximal Ideal)**.** Let $I, J \subset R$ be ideals. We say that I is a *maximal ideal* of R if whenever $I \subset J \subset R$, either $J = I$ or $J = R$.

Example 3.5. We claim that the ideal $(x_1, \ldots, x_n) \subset R$ is maximal. To see this, we suppose that $(x_1, \ldots, x_n)$ is not maximal. Then there exists an ideal J, where $(x_1, \ldots, x_n) \subset J \subset R$, such that $J \neq (x_1, \ldots, x_n)$ and $J \neq R$. Note that $(x_1, \ldots, x_n) \neq R$ since $1 \in R$ but $1 \notin (x_1, \ldots, x_n)$. Let $f \in J \setminus (x_1, \ldots, x_n)$. Then f cannot have terms that include any of the x_i's since if it did, then f would be in I. So $f = c$ for some $c \in k$ such that $c \neq 0$ (since if $c = 0$, then $c \in (x_1, \ldots, x_n)$). Then $f = c \in J$ which implies that

$\frac{1}{c} \cdot c = 1 \in J$. This contradicts the assumption that $J \neq R$. Hence, no such J exists and $(x_1, \ldots, x_n)$ is a maximal ideal in R.

Definition 3.6 (Prime Ideal)**.** Let $I \subset R$ be an ideal. Then I is *prime* if whenever $fg \in I$, either $f \in I$ or $g \in I$.

Lemma 3.7. *Maximal ideals are prime ideals.*

Proof. Let $M \subset R$ be a maximal ideal and let $fg \in M$ such that $f \notin M$. Then since M is maximal, we have that $M + (f) = (1)$. So there is $m \in M$ and $h \in R$ such that $m + hf = 1$. Now

$$g = 1 \cdot g = (m + hf)g = mg + hfg.$$

Since $m \in M$ and $fg \in M$, we obtain that mg and hfg are in M by the absorption property of ideals and since ideals are closed under addition, we have $mg + hfg = g \in M$. Hence, M is a prime ideal in R. $\square$

Lemma 3.8. *Any ideal generated by a subset of the variables of R is prime, i.e.,*

$$I = (x_{i_1}, \ldots, x_{i_r} \mid \{x_{i_1}, \ldots, x_{i_r}\} \subseteq \{x_1, \ldots, x_n\})$$

is a prime ideal of R for any subset $\{x_{i_1}, \ldots, x_{i_r}\} \subseteq \{x_1, \ldots, x_n\}$.

Proof. Let $I = (x_{i_1}, \ldots, x_{i_r}) \subseteq k[x_1, \ldots, x_n]$ be an ideal. We will to show that I is prime by showing that R/I is an integral domain. Let $a + I, b + I \in R/I$ such that $ab + I = 0 + I$ and $a + I \neq 0 + I$. We will use a proof by contradiction. Suppose that $b + I \neq 0 + I$. Then a and b have at least one term g and h respectively, not divisible by x_{i_j} for any x_{i_j} generating I. Write

$$a = g + \sum_{\alpha} c_\alpha x^\alpha \qquad \text{and} \qquad b = h + \sum_{\beta} c_\beta x^\beta$$

where $g, h \notin I$, $c_\alpha, c_\beta \in k[x_1, \ldots, x_n]$ and $x^\alpha = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$, $x^\beta = x_{i_1}^{b_1} \cdots x_{i_r}^{b_r}$ where each $a_i, b_j \geq 0$. Then

$$ab = gh + g \sum_{\beta} c_\beta x^\beta + h \sum_{\alpha} c_\alpha x^\alpha + (\sum_{\beta} c_\beta x^\beta)(\sum_{\alpha} c_\alpha x^\alpha).$$

By the absorption property of ideals, $g \sum_\beta c_\beta x^\beta$ and $h \sum_\alpha c_\alpha x^\alpha$ are in I and since I is closed under multiplication, $(\sum_\beta c_\beta x^\beta)(\sum_\alpha c_\alpha x^\alpha) \in I$. Since $g \notin I$ and $h \notin I$, we have that $gh \notin I$. But this contradicts the fact that $ab \in I$, since $ab + I = 0 + I$. So we must have that $b + I = 0 + I$ and hence R/I is an integral domain. It follows that I is a prime ideal in $k[x_1, \ldots, x_n]$. $\square$

We have ways to construct new ideals from existing ones. One of these new constructions is described in the following definition:

Definition 3.9 (Ideal Quotient)**.** Let I, J be ideals in R. Then the *ideal quotient* of I and J is defined by

$$I : J = \{f \in R \mid fj \in I \text{ for each } j \in J\}.$$

Lemma 3.10. *Let $I, J \subset R$ be ideals. Then the ideal quotient $I : J$ is an ideal.*

Proof. Let I and J be ideals in R. We will first show that $I : J \neq \emptyset$. Take $0 \in R$. Since I is an ideal we know that $0 \in I$. We have

$$0 = 0j \in I$$

for all $j \in J$. It follows that $0 \in I : J$. Now suppose that $f, g \in I : J$. Then $fj, gj \in I$ for all $j \in J$. Since I is an ideal, we know by Definition 3.1(2) that $fj - gj \in I$. So we have

$$fj - gj = (f - g)j \in I$$

for all $j \in J$. By the definition of the ideal quotient, $f - g \in I : J$. Finally let $h \in R$ and $f \in I : J$. Then $fj \in I$ for all $j \in J$. From Definition 3.1(3) we know that $h(fj) \in I$. Then

$$h(fj) = (hf)j \in I$$

for all $j \in J$. Hence $hf \in I : J$. From Definition 3.1 we can conclude that $I : J$ is an ideal. $\square$

The construction of the ideal quotient leads us to the following definition of the *associated primes* of an ideal I.

Definition 3.11 (Associated Prime)**.** Let $I \subset R$ be an ideal and $J = (f) \subset R$ a principal ideal. If $(I : J) = P$ is prime, then we say that P is an *associated prime* of I. We say that f is the *annihilator* of I and we denote the *set of associated primes* of I by $\text{Ass}(R/I)$.

Example 3.12. Let $I = (x_1x_2, x_1x_3, x_2x_3)$ be an ideal of R. Let $g_1 = x_1$, $g_2 = x_2$, and $g_3 = x_3$ be three polynomials in R. Now we construct the ideal quotients

- $I : (x_1) = (x_2, x_3)$
- $I : (x_2) = (x_1, x_3)$
- $I : (x_3) = (x_1, x_2)$.

We know from Lemma 3.8 that each of these ideals is prime. To simplify notation, we let $P_1 = (I : x_1)$, $P_2 = (I : x_2)$, and $P_3 = (I : x_3)$. Then $P_1, P_2, P_3 \in \text{Ass}(R/I)$. In fact these are all of the associated primes of I; that is, $\text{Ass}(R/I) = \{P_1, P_2, P_3\}$.

1. Primary Decomposition

Our goal in this section is to show that any proper ideal $I \subset R$ can be written as the finite intersection of *primary* ideals. The content of this section can be found in [**2**], [**11**], [**18**], and [**23**].

Definition 3.13 (Primary Ideal)**.** Let $Q \subset R$ be an ideal. We say that Q is *primary* if whenever $fg \in Q$, either $f \in Q$ or $g^m \in Q$ for some integer $m \geq 1$.

Lemma 3.14. *Prime ideals are primary.*

PROOF. Let $P \subset R$ be a prime ideal such that $fg \in P$. Since P is prime, we know that either $f \in P$ or $g \in P$. If $f \in P$, then we are done. So suppose that $f \notin P$. Then we know that $g \in P$ which implies that $g^m \in P$ for any integer $m \geq 1$. It follows that P is primary. $\square$

Definition 3.15 (Radical of I)**.** Let $I \subset R$ be an ideal and define

$$\sqrt{I} = \{f \in R \mid f^n \in I \text{ for some integer } n \geq 1\}.$$

We call $\sqrt{I}$ the *radical* of I.

Lemma 3.16. *Let $I \subset R$ be an ideal. Then $\sqrt{I}$ is an ideal.*

PROOF. Since I is an ideal, we know that $0^1 \in I$ so $0 \in \sqrt{I}$. Now let $f, g \in \sqrt{I}$. Then by definition $f^m, g^n \in I$ for some $m, n \geq 1$. Consider $k = m + n - 1 \geq 1$. We claim that $(f-g)^k \in I$. We use the Binomial Theorem to expand $(f-g)^k$ and we obtain

$$\begin{aligned} (f-g)^k &= \binom{k}{0}(-1)^k g^k + \binom{k}{1} f(-1)^{k-1} g^{k-1} + \cdots + \binom{k}{k} f^k \\ &= \binom{m+n-1}{0}(-1)^{m+n-1} g^{m+n-1} + \binom{m+n-1}{1} f(-1)^{m+n-2} g^{m+n-2} + \cdots + \binom{m+n-1}{m+n-1} f^{m+n-1}. \end{aligned}$$

Now since either $i \geq m$ or $m + n - 1 - i \geq n$ for each i, we have that every term of the expansion is either a multiple of f^m or g^n so $(f-g)^k \in I$. It follows by definition that $f - g \in \sqrt{I}$. Finally, suppose that $f \in \sqrt{I}$ and $h \in R$. Then $f^m \in I$ for some m. By the properties of R we know that $h^m \in R$ and by the absorption property of I we have $f^m h^m = (fh)^m \in I$. It follows that $fh \in \sqrt{I}$ and $\sqrt{I}$ is an ideal. $\square$

Definition 3.17 (P-primary Ideal)**.** If $Q \subset R$ is a primary ideal and $\sqrt{Q} = P$ for some prime ideal P, then we say that Q is *P-primary*.

Lemma 3.18. *If $P \subset R$ is a prime ideal, then $\sqrt{P} = P$.*

PROOF. We will use a proof by double inclusion. Let $f \in \sqrt{P}$. Then $f^m \in P$ for some integer m. We can write $f^m = f \cdot f^{m-1}$. Since P is prime, either $f \in P$ or $f^{m-1} \in P$. If $f \in P$, then we are done. If $f^{m-1} \in P$ then we can write $f^{m-1} = f \cdot f^{m-2}$. Again since P is prime, either $f \in P$ or $f^{m-2} \in P$. Continuing in this way, we deduce that $f \in P$ and we have the first inclusion. Now let $g \in P$. Then $g^m \in P$ for any integer m and hence $g \in \sqrt{P}$. So $\sqrt{P} \supseteq P$ and equality follows by double inclusion. $\square$

Lemma 3.19. *Let $Q \subset R$ be a primary ideal. Then $\sqrt{Q}$ is a prime ideal P. Furthermore, P is the smallest prime ideal of R containing Q.*

PROOF. Let Q be a primary ideal in R. We want to show that $\sqrt{Q}$ is prime. Suppose that $ab \in \sqrt{Q}$. Then $(ab)^m = a^m b^m \in Q$ for some integer m. Since Q is primary we have that either $a^m \in Q$ or $(b^m)^n = b^{mn} \in Q$ for some n. It follows by definition that $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$ and thus $\sqrt{Q}$ is prime.

To show that P is the smallest prime ideal containing Q suppose that there is some prime ideal $P' \subset R$ such that $P' \neq P$ and $Q \subseteq P'$. Taking the radical of Q and P' we obtain that $\sqrt{Q} \subseteq \sqrt{P'}$. Since Q is P-primary, we have $P = \sqrt{Q}$ and by Lemma 3.18 $\sqrt{P'} = P'$ so we have $P = \sqrt{Q} \subseteq \sqrt{P'} = P'$. So if P' is a prime ideal containing Q then P' must contain P and the result follows. $\square$

Lemma 3.20. *Let $Q_1, \ldots, Q_s$ be P-primary ideals in R. Then*

$$Q_1 \cap \cdots \cap Q_s$$

is P-primary.

Proof. We want to show that $\sqrt{Q_1 \cap \cdots \cap Q_s} = P$. First we will show that

$$\sqrt{Q_1 \cap \cdots \cap Q_s} = \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_s}$$

by double inclusion. Let $f \in \sqrt{Q_1 \cap \cdots \cap Q_s}$. Then $f^m \in Q_1 \cap \cdots \cap Q_s$ for some integer m. So $f^m \in Q_i$ for each i. This implies that $f \in \sqrt{Q_i}$ for each i and hence $f \in \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_s}$. So $\sqrt{Q_1 \cap \cdots \cap Q_s} \subseteq \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_s}$. Now suppose that $f \in \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_s}$. Then $f \in \sqrt{Q_i}$ for each i. By definition, $f^{n_i} \in Q_i$ for each i where n_i is an integer. So $f^{n_1} \cdots f^{n_s} = f^{n_1 + \cdots + n_s} \in Q_1 \cap \cdots \cap Q_s$. It follows that $f \in \sqrt{Q_1 \cap \cdots \cap Q_s}$ and we have the second inclusion.

Now we have

$$\begin{aligned} \sqrt{Q_1 \cap \cdots \cap Q_s} &= \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_s} \\ &= P \cap \cdots \cap P && \text{(since } \sqrt{Q_i} = P \text{ for each } i\text{)} \\ &= P \end{aligned}$$

So $Q_1 \cap \cdots \cap Q_s$ is P-primary. $\square$

Lemma 3.21. *Let $Q \subset R$ be a P-primary ideal and let $f \in R$.*

(1) *If $f \in Q$, then $\sqrt{(Q : (f))} = R$.*
(2) *If $f \notin Q$, then $\sqrt{(Q : (f))} = P$.*
(3) *If $f \notin P$, then $Q : (f) = Q$.*

Proof. (1) Let $f \in Q$. Then $fg \in Q$ for any $g \in R$ so it follows by definition that $Q : (f) = R$. Then $\sqrt{(Q : (f))} = \sqrt{R} = R$ and we are done.

(2) Suppose that $f \notin Q$. We will show that $\sqrt{(Q : (f))} = P$ by double inclusion. Let $g \in \sqrt{(Q : (f))}$. Then $g^m \in Q : (f)$ for some integer m. By definition, $g^m f \in Q$ but since $f \notin Q$ and Q is primary, we must have that $(g^m)^k = g^{mk} \in Q$ for some k. Then $g \in \sqrt{Q} = P$ since Q is P-primary. So we have our first inclusion. Now let $h \in P = \sqrt{Q}$. We have $h^m \in Q$ for some m. By the absorption property of ideals, $h^m f \in Q$ which implies that $h^m \in Q : (f)$. By definition, $h \in \sqrt{(Q : (f))}$ and we have the second inclusion.

(3) We always have $Q : (f) \supseteq Q$ so we only need to show that $Q : (f) \subseteq Q$. Let $g \in Q : (f)$. Then $gf \in Q \subseteq \sqrt{Q} = P$. By assumption, $f \notin P$ and P is prime so we must have that $g \in P = \sqrt{Q}$. We claim that $g \in Q$. Seeking a contradiction, suppose that $g \notin Q$. Since $Q \subseteq Q : (f)$, we have that $g \notin Q : (f)$ which implies that $gf \notin Q$. This contradicts the fact that $gf \in Q$ so we must have that $g \in Q$ and we have the second inclusion. $\square$

Lemma 3.22. *Suppose that $P \subset R$ is prime. If $P = P_1 \cap \cdots \cap P_s$ where each P_i is prime, then $P = P_i$ for some i.*

Proof. Suppose that P is a prime ideal in R such that

$$P = P_1 \cap \cdots \cap P_s$$

where each P_i is prime. Clearly, $P \subseteq P_i$ for each i so it is enough to show that $P \supseteq P_i$ for some i. We will do so by contradiction. Suppose that $P \subsetneq P_i$ for all i. Then for each i there is $f_i \in P_i$ such that $f_i \notin P$. So $f_1 \cdots f_s \in P_1 \cap \cdots \cap P_s = P$. Since P is prime, we must have that $f_i \in P$ for some i. This contradicts our assumption that $f_i \notin P$ for all i. Hence $P = P_i$ for some i. $\square$

Definition 3.23 (Irreducible Ideal)**.** Let $I \subset R$ be a proper ideal. We say that I is *irreducible* if whenever $J, K \subset R$ are ideals such that

$$I = J \cap K$$

we have $I = J$ or $I = K$.

In Lemma 3.24, we will show that any proper ideal can be written as the intersection of irreducible ideals. The proof uses the fact that R is a *Noetherian ring* which we define in Definition 4.1. The reader should review this definition before proceeding. The definition can also be found in [**11**].

Lemma 3.24. *Let $I \subset R$ be a proper ideal. Then we can write*

$$I = I_1 \cap \cdots \cap I_r$$

where each I_i is an irreducible ideal.

Proof. We will use a proof by contradiction. Suppose that there is a nonempty collection of ideals $\mathcal{I}$ that cannot be written as a finite intersection of irreducible ideals. Since R is Noetherian, there exists a maximal element I in $\mathcal{I}$. Since I is reducible, we can write $I = J \cap K$ for some ideals J, K such that $J \neq I$ and $K \neq I$. We always have $I \subset J$ and $I \subset K$ but since I is maximal in $\mathcal{I}$, we must have that $J \notin \mathcal{I}$ and $K \notin \mathcal{I}$. So J and K can be written as finite intersections of irreducible ideals say

$$J = J_1 \cap \cdots \cap J_s \quad \text{and} \quad K = K_1 \cap \cdots \cap K_t$$

where each J_ℓ, K_m are irreducible. Then

$$I = J \cap K = J_1 \cap \cdots \cap J_s \cap K_1 \cap \cdots \cap K_t.$$

So I can be written as a finite intersection of irreducible ideals. This contradicts the fact that $I \in \mathcal{I}$. So we must have that $\mathcal{I} = \emptyset$ and hence every ideal in R can be written as a finite intersection of irreducible ideals. $\square$

Lemma 3.25. *If $I \subset R$ is an irreducible ideal, then I is primary.*

PROOF. Suppose that I is irreducible and let $fg \in I$ such that $f \notin I$. Consider the ascending chain of ideals

$$I \subset I : (g) \subset I : (g^2) \subset \cdots .$$

Since the polynomial ring R is Noetherian, there exists an integer N such that

$$I : (g^N) = I : (g^{N+1}) = \cdots .$$

We claim that

$$I = (I + (g^N)) \cap (I + (f)).$$

We will prove this by double inclusion. It is easy to see that $I \subseteq I + (g^N)$ and $I \subset I + (f)$. So $I \subseteq (I + (g^N)) \cap (I + (f))$ and we have our first inclusion. Now suppose that $h \in (I + (g^N)) \cap (I + (f))$. Then since $h \in I + (g^N)$, we can write

$$h = i + h'g^N$$

for some $i \in I$ and $h' \in R$. Since $fg \in I$ we have that $g(I + (f)) \subseteq I$. In particular, $gh \in I$ so multiplying h by g, we have that $gh = gi + h'g^{N+1}$. Rearranging, we obtain

$$h'g^{N+1} = gh - gi$$

where $gh \in I$ and $gi \in I$. Hence $h'g^{N+1} \in I$. By definition, $h' \in I + (g^{N+1}) = I + (g^N)$ (by the ascending chain condition for ideals) which implies that $h'g^N \in I$. Now since $i \in I$ and $h'g^N \in I$, we obtain that $h = i + h'g^N \in I$ and we have our second inclusion. This completes the proof of our claim.

By assumption, I is irreducible, so $I = I + (g^N)$ or $I = I + (f)$. Since $f \notin I$, we must have that $I = I + (g^N)$ so $g^N \in I$. It follows that I is primary. □

Definition 3.26 ((Minimal) Primary Decomposition)**.** Let $I \subset R$ be an ideal. Then a *primary decomposition* of I is given by

$$I = \bigcap_{i=1}^{r} Q_i$$

where Q_i is primary for each i. We call the decomposition *minimal* (or *irredundant*) if

(1) $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$ and
(2) $Q_i \not\supset \bigcap_{j \neq i} Q_j$.

We say that each Q_i is a *primary component* of I.

Example 3.27. Consider the ideal $I = (x^2 - 1, x + y, z) \subseteq k[x, y, z]$. Then a primary decomposition of I is given by

$$I = \underbrace{(z, x - 1, y + 1, z)}_{Q_1} \cap \underbrace{(x + 1, y - 1, z)}_{Q_2}.$$

Now $\sqrt{Q_1} = Q_1$ and $\sqrt{Q_2} = Q_2$. It is easy to see that $Q_1 \not\subset Q_2$ and $Q_1 \not\supset Q_2$ so $\sqrt{Q_1} \neq \sqrt{Q_2}$. Since the given decomposition satisfies both (1) and (2) in Definition 3.26, we can conclude that $(x - 1, y + 1, z) \cap (x + 1, y - 1, z)$ is an irredundant primary decomposition of I.

Theorem 3.28 (Existence of Primary Decomposition)**.** *Every proper ideal $I \subset R$ can be written as a finite intersection of primary ideals.*

Proof. Let I be a proper ideal in R. By Lemma 3.24, we can write I as the finite intersection of irreducible ideals. So let

$$I = Q_1 \cap \cdots \cap Q_s$$

where each Q_i is irreducible. From Lemma 3.25 we know that irreducible ideals are primary, i.e., Q_i is primary for each i. Hence, I can be written as the finite intersection of primary ideals. $\square$

Lemma 3.29. *Let I be a proper ideal in R and let*

$$I = Q_1 \cap \cdots \cap Q_r$$

be a primary decomposition of I. For any $f \in R$

$$\sqrt{I : (f)} = \sqrt{(Q_1 : (f))} \cap \cdots \cap \sqrt{(Q_r : (f))}.$$

Proof. Let $f \in R$. We will first show that if $I = Q_1 \cap \cdots \cap Q_r$ is a primary decomposition of I then

$$(Q_1 \cap \cdots \cap Q_r) : (f) = (Q_1 : (f)) \cap \cdots \cap (Q_r : (f)). \tag{1.1}$$

Suppose that $g \in (Q_1 \cap \cdots \cap Q_r) : (f)$. Then $gf \in Q_1 \cap \cdots \cap Q_r$. So $gf \in Q_i$ and $g \in (Q_i : (f))$ for all i. It follows that $g \in (Q_1 : (f)) \cap \cdots \cap (Q_r : (f))$ and we have the first inclusion. Now suppose that $h \in (Q_1 : (f)) \cap \cdots \cap (Q_r : (f))$. Then $h \in (Q_i : (f))$ for all i. It follows that $hf \in Q_i$ for all i so $hf \in Q_1 \cap \cdots \cap Q_r$ and hence $h \in (Q_1 \cap \cdots \cap Q_r) : (f)$.

Now we want to show that

$$\sqrt{(Q_1 : (f)) \cap \cdots \cap (Q_r : (f))} = \sqrt{(Q_1 : (f))} \cap \cdots \cap \sqrt{(Q_r : (f))} \tag{1.2}$$

Suppose that $g \in \sqrt{(Q_1 : (f)) \cap \cdots \cap (Q_r : (f))}$. Then $g^m \in (Q_1 : (f)) \cap \cdots \cap (Q_r : (f))$ for some integer m. So $g^m \in (Q_i : (f))$ for each i and $g \in \sqrt{(Q_i : (f))}$ by definition. It follows that $g \in \sqrt{(Q_1 : (f))} \cap \cdots \cap \sqrt{(Q_r : (f))}$. Now let $h \in \sqrt{(Q_1 : (f))} \cap \cdots \cap \sqrt{(Q_r : (f))}$. Then $h \in \sqrt{(Q_i : (f))}$ for each i and moreover $h^{m_i} \in (Q_i : (f))$ for some integer m_i. We have that $(h^{m_1} \cdots h^{m_r}) = h^{m_1 + \cdots + m_r} \in (Q_1 : (f)) \cap \cdots \cap (Q_r : (f))$. By definition $h \in \sqrt{(Q_1 : (f)) \cap \cdots \cap (Q_r : (f))}$ and we have the second inclusion.

Now combining 1.1 and 1.2 with the fact that $I = Q_1 \cap \cdots \cap Q_r$, we have

$$\begin{aligned} \sqrt{I : (f)} &= \sqrt{(Q_1 \cap \cdots \cap Q_r) : (f)} \\ &= \sqrt{(Q_1 : (f)) \cap \cdots \cap (Q_r : (f))} \\ &= \sqrt{(Q_1 : (f))} \cap \cdots \cap \sqrt{(Q_r : (f))}. \end{aligned}$$

$\square$

Theorem 3.30 (First Uniqueness Theorem for Primary Decomposition)**.** *Every proper ideal $I \subset R$ has a minimal primary decomposition. If*

$$I = Q_1 \cap \cdots Q_r$$

is a minimal primary decomposition of I where each Q_i is P_i-primary, then

$$\{P_1, \ldots, P_r\} = \{\sqrt{I : (f)} \mid f \in R, \sqrt{I : (f)} \text{ is prime}\},$$

and hence the set of associated primes of I is independent of the decomposition of I.

Proof. Let I be a proper ideal in R. By Theorem 3.28, we know that we can write I as the intersection of finitely many primary ideals. If a primary ideal in the decomposition contains the intersection of the remaining primary ideals, then we can remove this ideal without changing the intersection. This guarantees (2) in Definition 3.26. By Lemma 3.20, we know that if $Q_{i_1}, \ldots, Q_{i_t}$ are P-primary ideals then $Q_{i_1} \cap \cdots \cap Q_{i_t}$ is P-primary. So if two or more primary ideals in the decomposition of I are P-primary for some P, we can replace these ideals with their intersection which guarantees (1) in Definition 3.26. Hence, I has a minimal primary decomposition.

We will show that $\{P_1, \ldots, P_r\} = \{\sqrt{I : (f)} \mid f \in R, \sqrt{I : (f)} \text{ is prime}\}$ by double inclusion. Suppose that $P \in \{P_1, \ldots, P_r\}$. Then $P = P_i = \sqrt{Q_i}$ for some i. Since the decomposition of I is minimal, there exists $f \in Q_1 \cap \cdots \cap \hat{Q}_i \cap \cdots \cap Q_r$ (where $\hat{Q}_i$ denotes the absence of Q_i) such that $f \notin Q_i$. Then by Lemma 3.29 we have

$$\sqrt{I : (f)} = \sqrt{Q_1 : (f)} \cap \cdots \cap \sqrt{Q_i : (f)} \cap \cdots \cap \sqrt{Q_r : (f)}. \tag{1.3}$$

Since $f \in Q_j$ for all $j \neq i$, we have by Lemma 3.21(1) that $\sqrt{Q_j : (f)} = R$ for all $j \neq i$. So 1.3 becomes

$$\sqrt{I : (f)} = R \cap \cdots \cap \sqrt{Q_i : (f)} \cap \cdots \cap R = \sqrt{Q_i : (f)}. \tag{1.4}$$

Now by Lemma 3.21(2) we have that since $f \notin Q_i$, $\sqrt{Q_i : (f)} = P_i$. So from 1.4, we obtain

$$\sqrt{I : (f)} = P_i. \tag{1.5}$$

It follows that $P_i \in \{\sqrt{I : (f)} \mid f \in R, \sqrt{I : (f)} \text{ is prime}\}$ and we have the first inclusion.

Now suppose that $P = \sqrt{I : (f)} \in \{\sqrt{I : (f)} \mid f \in R, \sqrt{I : (f)} \text{ is prime}\}$ for some $f \in R$. We know that f is not in I since if it were then $\sqrt{I : (f)} = R$ by Lemma 3.21(1). So there exists at least one Q_i appearing in the minimal decomposition of I such that $f \notin Q_i$. Relabel the primary components of I in the following way:

- $f \in Q_1, \ldots, Q_t$ and
- $f \notin Q_{t+1}, \ldots, Q_r$.

By Lemma 3.29 we can write

$$P = \sqrt{I : (f)} = \sqrt{Q_1 : (f)} \cap \cdots \cap \sqrt{Q_t : (f)} \cap \sqrt{Q_{t+1} : (f)} \cap \cdots \cap \sqrt{Q_r : (f)}. \tag{1.6}$$

Since $f \in Q_1, \ldots, Q_t$, we know by Lemma 3.21(1) that the first t components of 1.6 are equal to R. So we have

$$P = \sqrt{Q_{t+1} : (f)} \cap \cdots \cap \sqrt{Q_r : (f)} \tag{1.7}$$

and since $f \notin Q_{t+1}, \ldots, Q_r$, we know by Lemma 3.21(2) that $\sqrt{Q_i : (f)} = P_i$ for $i = t+1, \ldots, r$. So from 1.7 we obtain

$$P = P_{t+1} \cap \cdots \cap P_r.$$

Since each P_i is prime and P is prime, we can apply Lemma 3.22 to conclude that $P = P_i$ for some $i = t+1, \ldots, r$. It follows that $P \in \{P_1, \ldots, P_r\}$ and we have the second inclusion. □

Remark 3.31. It should be noted that the set of associated primes of an ideal I as defined in Definition 3.11 is exactly the set $\{P_1, \ldots, P_r\}$ defined in Theorem 3.30; that is, if $I \subset R$ has a minimal primary decomposition given by

$$I = Q_1 \cap \cdots \cap Q_r$$

where $\sqrt{Q_i} = P_i$ for each i, then

$$\{(I : (f)) \mid f \in R, (I : (f)) \text{ is prime}\} = \{P_1, \ldots, P_r\}.$$

The proof can be found in Proposition 8.22 of [**23**].

In the following example, we illustrate how the First Uniqueness Theorem for Primary Decomposition can enable us to find the set of associated primes of an ideal $I \subset R$.

Example 3.32. Let $I = (x_1, x_2x_3, x_2x_4, x_3x_4) \subset k[x_1, x_2, x_3, x_4]$ be an ideal. A minimal primary decomposition of I is given by

$$I = (x_1, x_2, x_3) \cap (x_1, x_2, x_4) \cap (x_1, x_3, x_4).$$

By Lemma 3.8 we know that $(x_1, x_2, x_3), (x_1, x_2, x_4)$, and (x_1, x_3, x_4) are prime ideals in $k[x_1, x_2, x_3, x_4]$ and by Lemma 3.18, $\sqrt{(x_1, x_2, x_3)} = (x_1, x_2, x_3)$, $\sqrt{(x_1, x_2, x_4)} = (x_1, x_2, x_4)$, and $\sqrt{(x_1, x_3, x_4)} = (x_1, x_3, x_4)$. So it follows from the First Uniqueness Theorem for Primary Decomposition that $\text{Ass}(R/I) = \{(x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_4)\}$.

2. Monomial Ideals

Throughout the rest of this project we will be concerned with properties of *monomial ideals*. In this section we will define various types of monomial ideals and characterize some of their properties.

Definition 3.33 ((Square-free) Monomial)**.** A polynomial of the form

$$f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

with each $\alpha_i \geq 0$ is called a *monomial* in R. A monomial is *square-free* if $\alpha_i = 0$ or 1 for all $i = 1, \ldots, n$.

Definition 3.34 ((Square-free) Monomial Ideal)**.** A *monomial ideal* $I \subset R$ is an ideal generated by a collection of monomials in R, i.e.,

$$I = (x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ a monomial in } R).$$

If I is generated by square-free monomials, then we say that I is a *square-free monomial ideal*.

Example 3.35. Consider the ideals $I = (x)$ and $J = (x, y^2)$ in $k[x, y]$. Since we can write

$$x = x^1y^0 \quad \text{and} \quad y^2 = x^0y^2,$$

both I and J are monomial ideals. Naturally, I is square-free.

Every square-free monomial ideal I has a dual square-free monomial ideal $I^\vee$ called the *Alexander dual* of I as defined in [**14**].

Definition 3.36 (Alexander Dual)**.** If $I = (x_{1,1}x_{1,2}\cdots x_{1,t_1}, \ldots, x_{s,1}x_{s,2}\cdots x_{s,t_s}) \subset k[x_1, \ldots, x_n]$ is a square-free monomial ideal, then the *Alexander dual* of I, denoted $I^\vee$, is the square-free monomial ideal

$$I^\vee = (x_{1,1}, \ldots, x_{1,t_1}) \cap \cdots \cap (x_{s,1}, \ldots, x_{s,t_s}).$$

Example 3.37. Let $J = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4) \subset k[x_1, x_2, x_3, x_4]$. Then by Definition 3.36, the Alexander dual of J is given by

$$J^\vee = (x_1, x_2, x_3) \cap (x_1, x_2, x_4) \cap (x_1, x_3, x_4).$$

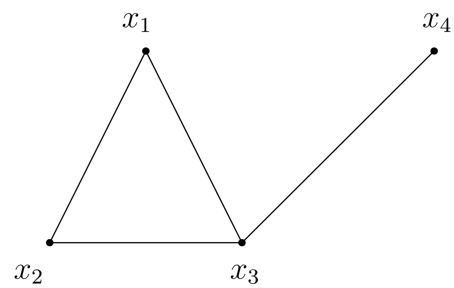
It is difficult in general to specify what the generators of $I^\vee$ will be directly from the generators of I. However, in Chapter 5 we will see that the ideal J given in Example 3.37 is of a special type and we will show which monomials generate $J^\vee$ in this case. The result will show that the Alexander dual $J^\vee$ in the above example is equal to the ideal $I = (x_1, x_2x_3, x_2x_4, x_3x_4)$ from Example 3.32.

2.1. Simple Graphs and Monomial Ideals. There exists a one-to-one correspondence between finite simple graphs in graph theory and monomial ideals in commutative algebra. This correspondence is developed by relabeling the vertices of a simple graph G with variables in the polynomial ring over a field k, $R = k[x_1, \ldots, x_n]$. We are interested in studying the ideals generated by the edges of G and their associated primes. The *edge ideal* was first introduced in [**25**].

Definition 3.38 (Edge Ideal)**.** Let G be a finite simple graph on the vertex set $V_G = \{x_1, \ldots, x_n\}$ with the edge set E_G. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k. Then the *edge ideal of* G, $I(G) \subset R$, is generated by the set of monomials x_ix_j such that $x_ix_j \in E_G$, i.e.,

$$I(G) = (x_ix_j \mid x_ix_j \in E_G).$$

Example 3.39. Let G be the connected graph in Example 2.5. By relabeling the vertices v_1, v_2, v_3 and v_4 with the variables x_1, x_2, x_3 and x_4 respectively, we can define the edge ideal of G in the polynomial ring $k[x_1, x_2, x_3, x_4]$ over an arbitrary field k. The relabeled graph G is shown below.



G

Using the edge set of G given by $E_G = \{x_1x_2, x_1x_3, x_2x_3, x_3x_4\}$, we obtain the edge ideal

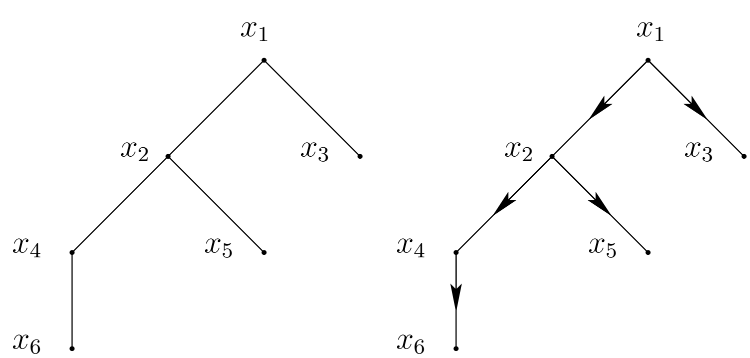
$$I(G) = (x_1x_2, x_1x_3, x_2x_3, x_3x_4).$$

In Definition 2.3 we defined a path of a graph G. By recognizing that an edge is a path of length one, we can extend the idea of an edge ideal to a *path ideal*. This concept was first introduced by Conca and De Negri in [**8**].

Definition 3.40 (Path Ideal)**.** Let $G = (V_G, E_G)$ be a finite simple graph. Then a *path ideal* of G is generated by the set of monomials $x_{i_1} \cdots x_{i_{t+1}}$ such that $x_{i_1}, \ldots, x_{i_{t+1}}$ is a path of G of length t. We denote the path ideal generated by paths of length t by $I_t(G)$.

We saw in Example 2.20 that the orientation of a graph G affects the number of paths of length t that appear in G. Naturally, the orientation of a graph also affects the generators of a path ideal as we will demonstrate in the following example.

Example 3.41. Suppose that Γ_1 and Γ_2 are the unrooted and rooted trees respectively from Example 2.20. Below we have relabeled the graphs with variables in $k[x_1, x_2, x_3, x_4, x_5, x_6]$.



Γ_1 Γ_2

Now suppose that we want to find I_3 corresponding to Γ_1 and Γ_2 respectively. We have

$$I_3(\Gamma_1) = (x_1x_2x_4x_6, x_3x_1x_2x_4, x_3x_1x_2x_5, x_5x_2x_4x_6)$$

and

$$I_3(\Gamma_2) = (x_1x_2x_4x_6).$$

It is easy to see that $I_3(\Gamma_1) \neq I_3(\Gamma_2)$. We have the containment $I_3(\Gamma_1) \supseteq I_3(\Gamma_2)$. However since $m = x_3x_1x_2x_4 \in I_3(\Gamma_1)$ and $x_1x_2x_4x_6$ does not divide m, the containment is proper; that is, $I_3(\Gamma_1) \not\subseteq I_3(\Gamma_2)$.

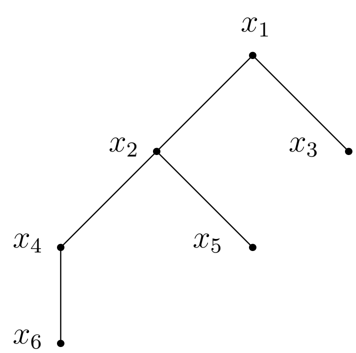
In Definition 3.36 we defined the Alexander dual of an arbitrary square-free monomial ideal I. If I_t is a path ideal corresponding to a graph G, then the Alexander dual of I_t is given by

$$I_t(G)^\vee = \bigcap (x_{i_1}, \ldots, x_{i_{t+1}})$$

where the intersection is over all paths $x_{i_1} \cdots x_{i_{t+1}}$ of G of length t.

Example 3.42. We saw in Example 3.41 that if Γ_1 is the unrooted tree shown below, then the 3-path ideal corresponding to Γ_1 is given by

$$I_3(\Gamma_1) = (x_1x_2x_4x_6, x_3x_1x_2x_4, x_3x_1x_2x_5, x_5x_2x_4x_6).$$



Γ_1

So by definition, the Alexander dual of $I_3(\Gamma_1)$ is

$$I_3(\Gamma_1)^\vee = (x_1, x_2, x_4, x_6) \cap (x_3, x_1, x_2, x_4) \cap (x_3, x_1, x_2, x_5) \cap (x_5, x_2, x_4, x_6).$$

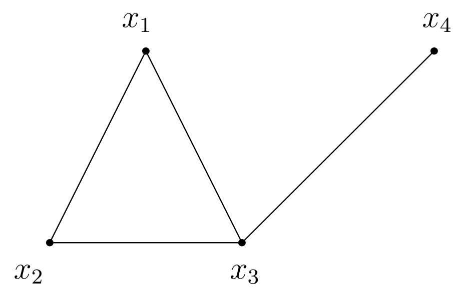
We conclude this chapter with an example that shows us how we can determine the associated primes of a graph using the edge ideal.

Example 3.43. In Example 3.39, we saw that the edge ideal of the graph G shown below is given by

$$I(G) = (x_1x_2, x_1x_3, x_2x_3, x_3x_4).$$

We claim that the set of associated primes of $I(G)$ corresponds to the set of all minimal vertex covers of G. We have that

$$\mathrm{Ass}(R/I(G)) = \{(x_1, x_3), (x_2, x_3), (x_1, x_2, x_4)\}.$$



G

Let $W_1 = \{x_1, x_3\}$, $W_2 = \{x_2, x_3\}$ and $W_3 = \{x_1, x_2, x_4\}$. We first need to check that each W_i is a vertex cover of G. Recall that if $G = (V_G, E_G)$, then $W \subseteq V_G$ is a vertex cover of G if W intersects every edge of G nontrivially (Definition 2.11). To simplify notation, let $e_1 = \{x_1, x_2\}$, $e_2 = \{x_1, x_3\}$, $e_3 = \{x_2, x_3\}$, and $e_4 = \{x_3, x_4\}$ be the sets of vertices corresponding to the edges of G. Then

$$W_1 \cap e_1 = \{x_1\},\ W_1 \cap e_2 = \{x_1, x_3\},\ W_1 \cap e_3 = \{x_3\},\ W_1 \cap e_4 = \{x_3\},$$
$$W_2 \cap e_1 = \{x_2\},\ W_2 \cap e_2 = \{x_3\},\ W_2 \cap e_3 = \{x_2, x_3\},\ W_2 \cap e_4 = \{x_3\},$$
$$W_3 \cap e_1 = \{x_1, x_2\},\ W_3 \cap e_2 = \{x_1\},\ W_3 \cap e_3 = \{x_2\},\ W_3 \cap e_4 = \{x_4\}.$$

Since $W_i \cap e_j \neq \emptyset$ for any i, j, we have that W_1, W_2, and W_3 are vertex covers of G. Next we need to check that these vertex covers are minimal. There are collectively seven distinct proper subsets of W_1, W_2, and W_3, namely $\{x_k\}$ where $k = 1, 2, 3, 4$, $\{x_1, x_2\}$, $\{x_1, x_4\}$, and $\{x_2, x_4\}$. We see that

$$\{x_1\} \cap e_3 = \emptyset,\ \{x_2\} \cap e_2 = \emptyset,\ \{x_3\} \cap e_1 = \emptyset,\ \{x_4\} \cap e_1 = \emptyset,$$
$$\{x_1, x_2\} \cap e_4 = \emptyset,\ \{x_1, x_4\} \cap e_3 = \emptyset, \text{ and } \{x_2, x_4\} \cap e_2 = \emptyset.$$

It follows that each W_i is a minimal vertex cover of G. Finally, we need to check that no other $W \subseteq V_G$ is a minimal vertex cover for G. There are six remaining possibities. They are $\emptyset, \{x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}$, and $\{x_1, x_2, x_3, x_4\}$. Since $\emptyset$ intersects every edge of G trivially, this cannot be a vertex cover of G. We have that

$$\{x_3, x_4\} \cap e_1 = \emptyset$$

so the above set does not satisfy the definition of a vertex cover. Now although the sets $\{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}$, and $\{x_1, x_2, x_3, x_4\}$ are vertex covers of G, they are not minimal since

$$W_1, W_2 \subset \{x_1, x_2, x_3\},\ W_1 \subset \{x_1, x_3, x_4\},\ W_2 \subset \{x_2, x_3, x_4\}, \text{ and}$$
$$W_1, W_2, W_3 \subset \{x_1, x_2, x_3, x_4\}.$$

So $\mathrm{Ass}(R/I(G))$ is the set of minimal vertex covers of G.

CHAPTER 4

The Cover Ideal and the Stabilization of $\mathrm{Ass}(R/J^s)$

We begin this chapter by defining a *Noetherian ring* [**11**], and the *index of stability* for the associated primes of ideals in a Noetherian ring. Many of the results discussed in this chapter are summarized in [**22**].

Definition 4.1 (Noetherian Ring)**.** A commutative ring R with identity is *Noetherian* if every ideal $I \subset R$ is finitely generated.

Theorem 4.2 (Hilbert Basis Theorem)**.** *Every ideal $I \subset k[x_1, \ldots, x_n]$ is finitely generated.*

The proof of Theorem 4.2 can be found in [**9**]. For our purposes, we need only to note that the polynomial ring $R = k[x_1, \ldots, x_n]$ is Noetherian by the Hilbert Basis Theorem.

Remark 4.3. It is important to note that if R is a commutative ring that satisfies the *ascending chain condition* for ideals; that is, if every ascending chain of ideals in R,

$$I_1 \subset I_2 \subset I_3 \subset \cdots,$$

eventually stabilizes, then R is *Noetherian.* This is equivalent to Definition 4.1.

Before giving the definition of the index of stability, we need to define the s^{th} *power* of an ideal I:

Definition 4.4 (s^{th} power of I)**.** Let R be a commutative ring and $I \subset R$ be an ideal. We define the s^{th} *power of I* by

$$I = (i_1 \cdots i_s \mid i_j \in I).$$

Example 4.5. Let $I = (x^2, y, xz^3) \subset k[x, y, z]$ be an ideal. Then the 2^{nd} and 3^{rd} powers of I are given by

$$I^2 = (x^4, x^2y, x^3z^3, y^2, xyz^3, x^2z^6)$$

and

$$I^3 = (x^6, x^4y, x^5z^3, x^2y^2, x^3yz^3, x^4z^6, y^3, xy^2z^3, x^2yz^6, x^3z^9),$$

respectively.

We are now ready to define the *index of stability.* In [**6**], Brodmann proved the following:

Theorem 4.6 (Brodmann, 1979)**.** *Let $I \subset R$ be an ideal where R is a commutative Noetherian ring. Then there exists $N \in \mathbb{N}$ such that*

$$\bigcup_{s=1}^{\infty} \mathrm{Ass}(R/I^s) = \bigcup_{s=1}^{N} \mathrm{Ass}(R/I^s).$$

We call the smallest such N the index of stability of $\mathrm{Ass}(R/I^s)$.

For an arbitrary ideal $I \subset R$, the index of stability N is difficult to compute. In [**20**], Hoa determined an upper bound on N for $\mathrm{Ass}(R/I^s)$ when $I \subset R$ is a monomial ideal.

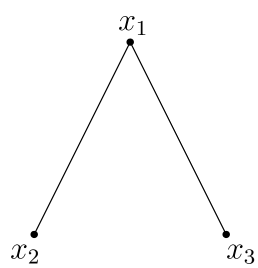
Theorem 4.7 (Hoa, 2006)**.** *If $I \subset R = k[x_1, \ldots, x_n]$ is a monomial ideal, then the index of stability N is bounded above by*

$$\max\{d(nr+r+d)(\sqrt{n})^{n+1}(\sqrt{2d})^{(n+1)(r-1)}, r(r+n)^4 r^{n+2} d^2 (2d^2)^{r^2-r+1}\}$$

where r is equal to the number of generators of I and d is the maximal degree of a generator.

By definition, if G is a finite simple graph, then the corresponding edge ideal $I(G)$ is a monomial ideal. We illustrate how large the upper bound on N corresponding to $\mathrm{Ass}(R/I(G)^s)$ may be in the following example:

Example 4.8. Let Γ be the tree below on the vertex set $V_\Gamma = \{x_1, x_2, x_3\}$ with the edge set $E_\Gamma = \{x_1x_2, x_1x_3\}$.



Γ

The edge ideal of Γ is given by

$$I(\Gamma) = (x_1x_2, x_1x_3).$$

So in this example $n = 3$, $r = 2$, and $d = 2$. Substituting these values into the upper bound formula given in Theorem 4.7, we obtain

$$\begin{aligned} N &\leq \max\{2(3\cdot 2+2+2)(\sqrt{3})^{3+1}(\sqrt{2\cdot 2})^{(3+1)(2-1)}, 2(2+3)^4 2^{3+2} 2^2 (2\cdot 2^2)^{2^2-2+1}\} \\ &= \max\{2880, 81920000\} \\ &= 81920000. \end{aligned}$$

In [**24**], Simis, Vasconcelos, and Villarreal proved that a graph G is bipartite if and only if $N = 1$ for $\mathrm{Ass}(R/I(G)^s)$. We proved in Corollary 2.28 that trees are bipartite and hence we know that if Γ is the tree given in Example 4.8, then we must have that $N = 1$. So the upper bound on N in this special case is much larger than the actual value of N.

Our goal in this chapter is to determine properties of $\mathrm{Ass}(R/(I(G)^\vee)^s)$ when G is a finite simple graph. In Chapter 3, we defined the Alexander dual of a path ideal as the intersection of the ideals generated by the vertices corresponding to the paths of length t. When $t = 1$, the path ideal is exactly the edge ideal defined in [**25**]. If G is a finite simple graph with corresponding edge ideal $I(G)$, we know what the generators of the Alexander dual $I(G)^\vee$ correspond to; they are minimal vertex covers (Definition 2.11) of the graph G.

Lemma 4.9. *Let $G = (V_G, E_G)$ be a finite simple graph with corresponding edge ideal $I(G)$. Then*

$$I(G)^\vee = (x_{i_1} \cdots x_{i_r} \mid \ W = \{x_{i_1}, \ldots, x_{i_r}\} \textit{ is a minimal vertex cover of } G).$$

We call $I(G)^\vee$ the cover ideal of G.

Proof. Let G be a finite simple graph on the vertex set $V_G = \{x_1, \ldots, x_n\}$ with the edge set $E_G = \{e_1, \ldots, e_t\}$. By definition we can write

$$I(G)^\vee = \bigcap_{x_j x_k \in E_G} (x_j, x_k).$$

So we need to show that

$$\bigcap_{x_j x_k \in E_G} (x_j, x_k) = (x_{i_1} \cdots x_{i_r} \mid \ W = \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a minimal vertex cover of } G).$$

We prove this by double inclusion. First suppose that M is a monomial in $\bigcap_{x_j x_k \in E_G} (x_j, x_k)$. Then $M \in (x_j, x_k)$ for each j, k. This implies that M is divisible by at least one of x_j or x_k for each $x_j x_k \in E_G$. Suppose without loss of generality, that M is divisible by exactly one vertex in each edge, say $x_{i_1}, \ldots, x_{i_t}$ where each $x_{i_\ell} \in e_\ell$. Then we can write

$$M = x_{i_1} \cdots x_{i_t} m$$

where m is a monomial in $k[x_1, \ldots, x_n]$. We claim that the set

$$W = \{x_{i_1}, \ldots, x_{i_t}\}$$

is a minimal vertex cover of G. Since $W \cap e_\ell = \{x_{i_\ell}\}$ for each ℓ; that is, W intersects each edge nontrivially, we know that W is a vertex cover of G. To show that W is minimal, let W' be a proper subset of W. Suppose that $x_{i_\ell} \in W \setminus W'$ for some $1 \leq \ell \leq t$. Then $W' \cap e_\ell = \emptyset$ so W' is not a vertex cover of G. Since W' was arbitrary, this is true for any proper subset of W. Hence W is a minimal vertex cover of G. We have that

$$M = x_{i_1} \ldots x_{i_t} m$$

where $W = \{x_{i_1}, \ldots, x_{i_t}\}$ is a minimal vertex cover of G so

$$M \in (x_{i_1} \cdots x_{i_r} \mid \ W = \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a minimal vertex cover of } G)$$

and we have the first inclusion. Now let M' be a monomial such that

$$M' \in (x_{i_1} \cdots x_{i_r} \mid \ W = \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a minimal vertex cover of } G).$$

Then we can write

$$M' = x_{i_1} \cdots x_{i_r} m'$$

where m' is a monomial in $k[x_1, \ldots, x_n]$ and $W = \{x_{i_1}, \ldots, x_{i_r}\}$ is a minimal vertex cover of G. By definition, $W \cap e_\ell \neq \emptyset$ for each $1 \leq \ell \leq t$. So for each $x_j x_k \in E_G$, at least one of x_j or x_k divides M'. This implies that $M' \in (x_j, x_k)$ for each j, k such that $x_j x_k \in E_G$. It follows that

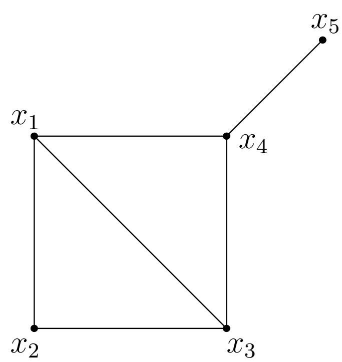
$$M' \in \bigcap_{x_j x_k \in E_G} (x_j, x_k)$$

and we have the second inclusion. This completes the proof. □

We illustrate the result of Lemma 4.9 in the following example:

Example 4.10. Let G be the graph shown below. The edge ideal of G is given by

$$I(G) = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_3x_4, x_4x_5).$$



G

Then by Definition 3.36 the Alexander dual of $I(G)$ is

$$I(G)^\vee = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5).$$

After computing the intersection, we obtain

$$I(G)^\vee = (x_1x_3x_5, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4).$$

In Example 2.12, we saw that $\{x_1, x_3, x_5\}$ is a minimal vertex cover of G so the first generator of $I(G)^\vee$ corresponds to a minimal vertex cover as desired. Let $W_1 = \{x_1, x_2, x_4\}$, $W_2 = \{x_1, x_3, x_4\}$, and $W_3 = \{x_2, x_3, x_4\}$. It is easy to check that every edge in our graph intersects each W_i nontrivially and hence W_1, W_2, and W_3 are vertex covers of G. Now we claim that no proper subset of each W_i is a vertex cover of G. We first check that no single vertex $\{x_j\}$ forms a vertex cover for G. We have $\{x_j\} \cap \{x_4, x_5\} = \emptyset$ for $j = 1, 2, 3$ and $\{x_j\} \cap \{x_1, x_2\} = \emptyset$ for $j = 4, 5$ where $x_4x_5, x_1x_2 \in E_G$. Finally, we will show that any $\{x_j, x_k\} \subset W_i$ for some i is not a vertex cover of G. There are six distinct proper subsets of W_1, W_2 and W_3, namely $W_1' = \{x_1, x_2\}$, $W_2' = \{x_1, x_3\}$, $W_3' = \{x_2, x_3\}$, $W_4' = \{x_1, x_4\}$, $W_5' = \{x_2, x_4\}$, and $W_6' = \{x_3, x_4\}$. The intersection of the subsets W_1', W_2', and W_3' with

$\{x_4, x_5\}$, where x_4x_5 is an edge of G, is empty and hence W_i' for $i = 1, 2, 3$ are not vertex covers of G. We also have that $W_4' \cap \{x_2, x_3\} = W_5' \cap \{x_1, x_3\} = W_6' \cap \{x_1, x_2\} = \emptyset$ where $x_2x_3, x_1x_3, x_1x_2 \in E_G$. So W_i' is not a vertex cover of G for any $i = 1, \ldots, 6$. Since we have checked that no proper subset of W_1, W_2, and W_3 are vertex covers of G, we can conclude that the generators of $I(G)^\vee$ correspond to minimal vertex covers of G.

To simplify notation, we will let $J = I(G)^\vee$ when the graph G to which the ideal is associated is clear. Recall from Chapter 3, we proved the First Uniqueness Theorem for Primary Decomposition (Theorem 3.30). We note that the Alexander dual of an ideal $I \subset R$ is defined in terms of its primary decomposition. Moreover, the ideals appearing in the intersection $I^\vee$ are prime by Lemma 3.8. It follows from Theorem 3.30 that the ideals appearing in the intersection defining $I^\vee$ are the associated primes of $I^\vee$. So given a graph G with cover ideal J, we know which ideals make up $\text{Ass}(R/J)$.

Example 4.11. Let Γ be the tree in Example 4.8 on the vertex set $V_\Gamma = \{x_1, x_2, x_3\}$ with the edge set $E_\Gamma = \{x_1x_2, x_1x_3\}$. The minimal vertex covers of Γ are given by $W_1 = \{x_1\}$ and $W_2 = \{x_2, x_3\}$. So if J is the ideal generated by the minimal vertex covers of Γ (the cover ideal), then $J = (x_1, x_2x_3)$. By Lemma 4.9, J is the Alexander dual of $I(\Gamma)$ so we can write

$$J = (x_1, x_2) \cap (x_1, x_3).$$

We can see that this is a primary decomposition of J and since (x_1, x_2) and (x_1, x_3) are prime ideals, $\text{Ass}(R/J) = \{(x_1, x_2), (x_1, x_3)\}$.

Given a finite simple graph G, we know which prime ideals make up $\text{Ass}(R/J)$. However, in order to better understand the behaviour of $\text{Ass}(R/J^s)$ for $s \geq 2$, we focus on special families of graphs. In particular, we will show that if G is a tree, then the index of stability N is equal to 1. To do this, we first need the following definition:

Definition 4.12 (Perfect Graph)**.** A graph G is *perfect* if for every induced subgraph H of G, $\omega(H) = \chi(H)$.

Recall that for a graph G, $\omega(G)$ denotes the *clique number* (Definition 2.16) and $\chi(G)$ the *chromatic number* (Definition 2.30) of the graph.

Lemma 4.13. *Let $G = (V_G, E_G)$ be a finite simple graph. If G is bipartite, then G is a perfect graph.*

Proof. Let $G = (V_G, E_G)$ be a bipartite graph and let H be an induced subgraph of G on the vertex set $V_H \subset V_G$ with the edge set E_H. Then H is either a set of isolated vertices or a bipartite graph. If H is a set of isolated vertices, then H is 1-colourable since $E_H = \emptyset$ so we can colour every vertex in V_H with the same colour. Trivially, $\chi(H) = 1$ and since a vertex is a clique of size one and H has no edges $\omega(H) = 1$. So in this case $\chi(H) = 1 = \omega(H)$. Now if H is bipartite, then we know by Theorem 2.33 that $\chi(H) = 2$ so we want to show that $\omega(H) = 2$. Suppose that H contains an induced clique $\mathcal{K}_n$ for some $n \geq 3$. Then we have two cases: n is odd and n is even. If n is odd, then H would contain an odd cycle as an induced subgraph. This would contradict the assumption that H is bipartite (Theorem 2.26). So suppose that n is even. Let $V_{\mathcal{K}_n} = \{x_{i_1}, \ldots, x_{i_{2k}}\} \subset V_H$ denote the vertex set of the induced clique $\mathcal{K}_n$. By Definition 2.15, we know that every pair of vertices $x_{i_j}, x_{i_\ell} \in V_{\mathcal{K}_n}$ is adjacent. Without loss of generality, consider the vertices $x_{i_1}, x_{i_2}, x_{i_3} \in V_{\mathcal{K}_n}$. We have that $x_{i_1}x_{i_2}$, $x_{i_2}x_{i_3} \in E_{\mathcal{K}_n}$, where $E_{\mathcal{K}_n}$ denotes the edge set of the induced clique $\mathcal{K}_n$. Moreover $x_{i_3}x_{i_1} \in E_{\mathcal{K}_n}$, but then

$$x_{i_1}x_{i_2}x_{i_3}x_{i_1}$$

is a cycle of length 3 in $\mathcal{K}_n$. This would imply that H contains an odd induced cycle contradicting the assumption that H is bipartite. Hence, the largest induced clique of H is of size 2; that is, $\omega(H) = 2$. Since for every induced subgraph H of G we have that $\chi(H) = \omega(H)$, we can conclude that G is a perfect graph. □

In [**13**], Francisco, Hà, and Van Tuyl prove the following theorem.

Theorem 4.14. *Let G be a simple graph and let J be the cover ideal of G. Then the following are equivalent:*

(1) *G is a perfect graph.*
(2) *For all $s \geq 1$, $P = (x_{i_1}, \ldots, x_{i_r}) \in \text{Ass}(R/J^s)$ if and only if the induced graph on $\{x_{i_1}, \ldots, x_{i_r}\}$ is a clique of size $1 < r \leq s+1$ in G.*

In Lemma 4.13 we showed that any bipartite graph is a perfect graph so using this result and Theorem 4.14, we are able to prove the following:

Theorem 4.15. *Let G be a tree with cover ideal J. Then*

$$\text{Ass}(R/J) = \text{Ass}(R/J^s)$$

for all integers $s \geq 1$.

Proof. Let G be a tree with cover ideal J. We will prove that $\text{Ass}(R/J) = \text{Ass}(R/J^s)$ for all $s \geq 1$ by double inclusion. Let $P \in \text{Ass}(R/J)$. By Lemma 4.9, we know that $P = (x_i, x_j)$ where $x_ix_j \in E_G$. Now by Corollary 2.28, we know that trees are bipartite so it follows from Lemma 4.13 that G is a perfect graph and we can apply Theorem 4.14. Since $P = (x_i, x_j) \in \text{Ass}(R/J^1)$, we have that $\{x_i, x_j\}$ is a clique of size 2. Now since $s \geq 1$, we obtain that $s + 1 \geq 2$. Applying Theorem 4.14 again, we can conclude that $P \in \text{Ass}(R/J^s)$ for all $s \geq 1$ and we have the first inclusion. Suppose now that $P = (x_{i_1}, \ldots, x_{i_r}) \in \text{Ass}(R/J^s)$ for all $s \geq 1$. By Theorem 4.14, $\{x_{i_1}, \ldots, x_{i_r}\}$ is a clique of size $1 < r \leq s + 1$. Since G is a tree, we know by Lemma 2.21 that the size of the largest clique of G is 2, i.e., $\omega(G) = 2$. So we must have that $r = 2$. Now $P = (x_{i_1}, x_{i_2})$ where $\{x_{i_1}, x_{i_2}\}$ is a clique of size $2 \leq s + 1$, so from Theorem 4.14 we can conclude that $P \in \text{Ass}(R/J^1)$. This completes the proof. □

CHAPTER 5

Stars and Associated Primes of J_2^s

We are interested in studying the associated primes of the Alexander dual of the path ideal $I_t(G)$ for paths that are longer than a single edge. In this chapter we study the associated primes of the dual of $I_2(G)$ in the case that our graph is a *star*.

Recall the definition of the Alexander dual of a path ideal:

Definition 5.1 (Alexander Dual of $I_t(G)$)**.** Let $G = (V_G, E_G)$ be a finite simple graph. Then the *Alexander dual* of $I_t(G)$ is given by

$$I_t(G)^\vee = \bigcap_{\{x_{i_1},\ldots,x_{i_{t+1}}\}} (x_{i_1}, \ldots, x_{i_{t+1}})$$

where $x_{i_1}, \ldots, x_{i_{t+1}}$ is a path of length t.

Throughout the rest of this chapter, let $J_t(G) = I_t(G)^\vee$. In Chapter 2, we defined a complete bipartite graph, $K_{m,n}$ (Definition 2.24). Here we consider $K_{1,n}$ as defined in [**10**].

Definition 5.2 (Star)**.** Let $K_{m,n}$ be a complete bipartite graph such that $m = 1$. We call $K_{1,n}$ a *star* and we say that the vertex in the singleton partition class is the star's *centre*.

We are interested in the associated primes of $(J_2(K_{1,n}))^s$ as s varies. Note that if we intend to study the properties of paths of length two, it is necessary that we treat a star as an unrooted tree since $K_{1,n}$ contains no directed paths of length greater than one. We are able to define the generators of the Alexander dual J_2 when our graph is a star:

Lemma 5.3. *Let $K_{1,n}$ be a star on the vertex set $\{z, x_1, \ldots, x_n\}$ with centre z. Then*

$$J_2 = (z) + (x_1 \cdots \hat{x}_i \cdots x_n \mid i = 1, \ldots, n)$$

where $\hat{x}_i$ denotes the absence of vertex x_i.

Proof. By definition we have

$$J_2 = \bigcap_{1\le j<k\le n} (x_j, z, x_k).$$

So we want to show that

$$\bigcap_{1\le j<k\le n} (x_j, z, x_k) = (z) + (x_1 \cdots \hat{x}_i \cdots x_n \mid i = 1, \ldots, n).$$

We proceed by double inclusion. Suppose that m is a monomial in $\bigcap_{1\le j<k\le n}(x_j, z, x_k)$. Then for each $1 \le j < k \le n$, we have $m \in (x_j, z, x_k)$. We have two cases: either $z \mid m$ or $z \nmid m$. If $z \mid m$, then $m \in (z)$ and hence $m \in (z) + (x_1 \cdots \hat{x}_i \cdots x_n \mid i = 1, \ldots, x_n)$. On the other hand, if $z \nmid m$ then we claim that $x_1 \cdots \hat{x}_i \cdots x_n \mid m$ for some i; that is, exactly one x_i does not divide m. We will show this by contradiction. Suppose that $x_j \nmid m$ and $x_k \nmid m$ for some $j < k$. We have assumed that $z \nmid m$ so $m \notin (x_j, z, x_k)$. This contradicts the fact that $m \in (x_j, z, x_k)$ for all $1 \le j < k \le n$. So $x_1 \cdots \hat{x}_i \cdots x_n \mid m$ for some i. It follows that $m \in (x_1 \cdots \hat{x}_i \cdots x_n \mid i = 1, \ldots, n)$ and we have the first inclusion.

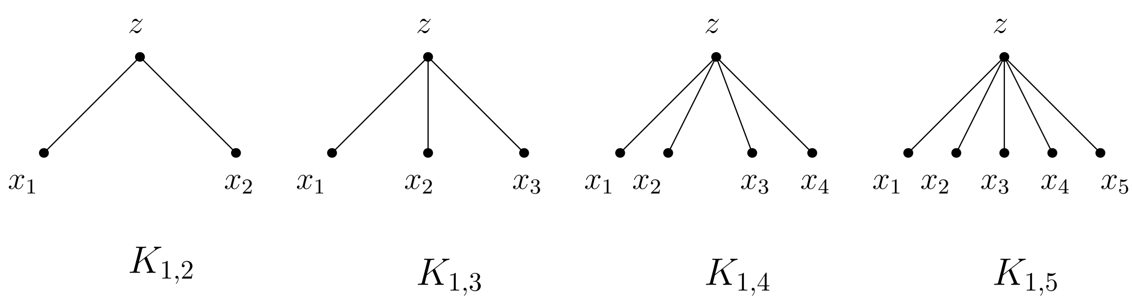
Now suppose that $m \in (z) + (x_1 \cdots \hat{x}_i \cdots x_n \mid i = 1, \ldots, n)$. Again we have two possibilities, either $z \mid m$ or $z \nmid m$. If $z \mid m$, then $m \in (x_j, z, x_k)$ for all $1 \le j < k \le n$. So $m \in \bigcap_{1\le j<k\le n}(x_j, z, x_k)$. Now if $z \nmid m$ then m is a multiple of some $x_1 \cdots \hat{x}_i \cdots x_n$. That is, we can write

$$m = x_1 \cdots \hat{x}_i \cdots x_n M$$

where M is a monomial in $k[z, x_1, \ldots, x_n]$. Now m is divisible by every x_ℓ except for when $\ell = i$. So possibly $m \notin (x_i, z, x_k)$ or $m \notin (x_j, z, x_i)$ for some $k > i$ or $j < i$. But m is divisible by every x_ℓ where $\ell \neq i$ so in particular $x_k \mid m$ and $x_j \mid m$ when $k > i$ and $j < i$. So $m \in (x_j, z, x_k)$ for all $1 \le j < k \le n$. Hence, $m \in \bigcap_{1\le j<k\le n}(x_j, z, x_k)$ and we have the second inclusion. Since $J_2 = \bigcap_{1\le j<k\le n}(x_j, z, x_k)$, the result follows by double inclusion. $\square$

Example 5.4. Below we have $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, and $K_{1,5}$, respectively. The centre of each of the stars shown is z. Here we list the corresponding Alexander dual, J_2 of each $K_{1,n}$ as described in Lemma 5.3:

- $K_{1,2} \to J_2 = (z, x_1, x_2)$
- $K_{1,3} \to J_2 = (z, x_1x_2, x_1x_3, x_2x_3)$
- $K_{1,4} \to J_2 = (z, x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4)$
- $K_{1,5} \to J_2 = (z, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4x_5, x_1x_3x_4x_5, x_2x_3x_4x_5)$.



Our goal is to prove Theorem 5.8. In order to do this we need the following lemma:

Lemma 5.5. *Let $K_{1,n}$ be a star on the vertex set $\{z, x_1, \ldots, x_n\}$ with centre z and corresponding Alexander dual J_2. If $J_2^s : (m) = (z, x_1, \ldots, x_n)$ where $m = z^e m'$ is a monomial in $k[z, x_1, \ldots, x_n]$ such that $z \nmid m'$, then*

$$m \mid z^e(x_1 \cdots x_n)^{s-e-1}.$$

PROOF. Let m be a monomial in $k[z, x_1, \ldots, x_n]$ such that $J_2^s : (m) = (z, x_1, \ldots, x_n)$, $m = z^e m'$, and $z \nmid m'$. Since $z \in (z, x_1, \ldots, x_n)$, we have that $zm \in J_2^s$. So we can write

$$zm = m_1 \cdots m_s M \tag{0.1}$$

where each m_j is a minimal generator of J_2 and M is a monomial in $k[z, x_1, \ldots, x_n]$. Now $m = z^e m'$ by assumption so substituting this expression into 0.1 we have

$$z^{e+1} m' = m_1 \cdots m_s M.$$

We know that z does not divide M since if it did, we would have

$$z^e m' = m_1 \cdots m_s (\frac{M}{z}),$$

so $z^e m' = m$ would be a multiple of a generator of J_2^s; that is, $m \in J_2^s$. This would contradict the fact that $J_2^s : (m) \neq (1)$. Hence, exactly $e + 1$ of the m_j's are equal to z. Without loss of generality, suppose that $m_1 = m_2 = \cdots = m_{e+1} = z$. Then we can write

$$z^{e+1} m' = z^{e+1} m_{e+2} \cdots m_s M. \tag{0.2}$$

Cancelling z^{e+1} on both sides of 0.2, we obtain

$$m' = m_{e+2} \cdots m_s M. \tag{0.3}$$

We claim that each x_i appears at most $s - e - 1$ times on the right-hand side of 0.3. We will prove this by contradiction. Suppose that x_i^{s-e} divides m' for some i. Since $x_i \in (z, x_1, \ldots, x_n)$, we have $x_i m = x_i z^e m' \in J_2^s$. So we can write

$$x_i z^e m' = \ell_1 \cdots \ell_s L \tag{0.4}$$

where each ℓ_j is a minimal generator of J_2 and L is a monomial in $k[z, x_1, \ldots, x_n]$. Now we can see that x_i does not divide L since if it did then we would have

$$z^e m' = \ell_1 \cdots \ell_s (\frac{L}{x_i}) \in J_2^s$$

which would contradict the fact that $J_2^s : (m) \neq (1)$. Since we have assumed that $x_i^{s-e} \mid m'$, we know that x_i must appear at least $s - e + 1$ times on the left-hand side of 0.4. This implies that x_i appears at least $s - e + 1$ times on the right-hand side of 0.4. We know that $x_i \nmid L$ so x_i must appear in at least $s - e + 1$ of the ℓ_j's. Without loss of generality, suppose that x_i appears in ℓ_j for $j = 1, \ldots, s - e + 1$. Then grouping these terms together we have

$$x_i z^e m' = (\ell_1 \cdots \ell_{s-e+1}) \ell_{s-e+2} \cdots \ell_s L. \tag{0.5}$$

We know that $x_i \mid \ell_j$ for $j = 1, \ldots s - e + 1$ so in particular $x_i \mid \ell_1$. Dividing by x_i on both sides of 0.5, we obtain

$$z^e m' = (\frac{\ell_1}{x_i} \ell_2 \cdots \ell_{s-e+1}) \ell_{s-e+2} \cdots \ell_s L. \tag{0.6}$$

Now z appears e times on the left hand-side of 0.6 so z must appear e times on the right-hand side. In Lemma 5.3 we showed that the minimal generators of J_2 take on either of the following two forms: z or $x_1 \cdots \hat{x_k} \cdots x_n$, where $\hat{x_k}$ denotes the absence of vertex x_k.

Since $x_i \mid \ell_j$ for $j = 1, \ldots, s-e+1$ and each ℓ_j is a minimal generator of J_2, we know that these generators are only in terms of the x_i's; that is, $z \neq \ell_j$ for $j = 1, \ldots, s-e+1$. So only the remaining $s-(s-e+1) = e-1$ ℓ_j's can be equal to z. Since z appears e times on the left-hand side of 0.6 and can appear at most $e-1$ times in the ℓ_j's, we must have that $z \mid L$; that is, $L = zK$ for some monomial K. Then

$$z^e m' = (\frac{\ell_1}{x_i}\ell_2 \cdots \ell_{s-e+1})\ell_{s-e+2} \cdots \ell_s zK.$$

Now $z^e m'$ is a multiple of s generators of J_2, namely $\ell_2, \ldots, \ell_s$ and z so $m = z^e m' \in J_2^s$. This contradicts the fact that $J_2^s : (m) \neq (1)$ so our assumption that $x_i^{s-e} \mid m'$ was false. It follows that each x_i appears at most $s-e-1$ times on the right-hand side of 0.3 and hence

$$m \mid z^e(x_1 \cdots x_n)^{s-e-1}.$$

$\square$

Given a star $K_{1,n}$ with corresponding Alexander dual J_2, we can find the annihilator of J_2^s that gives us the maximal ideal $(z, x_1, \cdots, x_n)$ in $k[z, x_1, \ldots, x_n]$.

Theorem 5.6. *Let $K_{1,n}$ be a star on the vertex set $\{z, x_1, \ldots, x_n\}$ with centre z and corresponding Alexander dual J_2. Then*

(1) *$J_2^s : (x_1^{n-2} \cdots x_n^{n-2}) = (z, x_1, \ldots, x_n)$ for $s = n-1$ and*
(2) *$J_2^s : (zx_1^{n-2}x_2^{s-2} \cdots x_n^{s-2}) = (z, x_1, \ldots, x_n)$ for $s > n-1$*

where $(z, x_1, \ldots, x_n) \subset k[z, x_1, \ldots, x_n]$ is a maximal ideal.

Proof.

(1) We first want to show that $x_1^{n-2} \cdots x_n^{n-2} \notin J_2^s$ when $s = n-1$. We note that every generator of J_2^s that is in terms of only the x_i's has degree $s(n-1)$. When $s = n-1$, such generators have degree $(n-1)^2 = n^2 - 2n + 1$. Now $x_1^{n-2} \cdots x_n^{n-2}$ has degree $n(n-2) = n^2 - 2n < (n-1)^2$. So we cannot have $x_1^{n-2} \cdots x_n^{n-2} \in J_2^s$. (Every other generator is some multiple of z so it is immediately apparent that these generators cannot divide $x_1^{n-2} \cdots x_n^{n-2}$.) Since $x_1^{n-2} \cdots x_n^{n-2} \notin J_2^s$ when $s = n-1$, we can conclude that $J_2^s : (x_1^{n-2} \cdots x_n^{n-2}) \neq (1)$.

Next we want to show that $(z, x_1, \ldots, x_n) \subset J_2^s : (x_1^{n-2} \cdots x_n^{n-2})$ for $s = n-1$. We proceed by showing that each generator of $(z, x_1, \ldots, x_n)$ is in $J_2^s : (x_1^{n-2} \cdots x_n^{n-2})$. Consider $z(x_1^{n-2} \cdots x_n^{n-2}) = z(x_1^{s-1} \cdots x_n^{s-1}) = z(x_1 \cdots x_{n-1})^{s-1}x_n^{s-1}$. Now $(x_1 \cdots x_{n-1})^{s-1} \in J_2^{s-1}$ and by the properties of ideals $(x_1 \cdots x_{n-1})^{s-1}x_n^{s-1} \in J_2^{s-1}$. Since $z \in J_2$, we have that $z(x_1 \cdots x_{n-1})^{s-1}x_n^{s-1} \in J_2^s$ and hence

$$z \in J_2^s : (x_1^{s-1} \cdots x_n^{s-1}).$$

Since $(x_1^{s-1} \cdots x_n^{s-1}) = (x_1^{n-2} \cdots x_n^{n-2})$, we have $z \in J_2^s : (x_1^{n-2} \cdots x_n^{n-2})$.

Next consider $x_i(x_1^{n-2}\cdots x_n^{n-2}) = x_i(x_1^{s-1}\cdots x_n^{s-1})$ for some i. We can write

$$x_i(x_1^{s-1}\cdots x_n^{s-1}) = \prod_{j=1, j\neq i}^{n} x_1\cdots\hat{x}_j\cdots x_n.$$

Note that there are exactly $n-1=s$ terms in this product and $x_1\cdots\hat{x}_j\cdots x_n \in J_2$ for each j, so $x_i(x_1^{s-1}\cdots x_n^{s-1}) \in J_2^s$. It follows that $x_i \in J_2^s : (x_1^{n-2}\cdots x_n^{n-2})$ for each i and hence $(z, x_1, \cdots, x_n) \subset J_2^s : (x_1^{n-2}\cdots x_n^{n-2})$ for $s = n-1$. We have shown that $J_2^s : (x_1^{n-2}\cdots x_n^{n-2}) \neq (1)$ and $(z, x_1, \ldots, x_n) \subset J_2^s : (x_1^{n-2}\cdots x_n^{n-2})$ so by the definition of a maximal ideal, we have that $J_2^s : (x_1^{n-2}\cdots x_n^{n-2}) = (z, x_1, \ldots, x_n)$.

(2) We will show that $J_2^s : (zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) \neq (1)$ for $s > n-1$ by contradiction. Suppose that $zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2} \in J_2^s$. Then we can write

$$zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2} = m_1\cdots m_s M \tag{0.7}$$

where each m_ℓ is a minimal generator of J_2 and M is a monomial in $k[z, x_1, \ldots, x_n]$.

Now we consider two cases. In the first case, $z \mid m_\ell$ for some ℓ which implies that $z = m_\ell$ by Lemma 5.3. Without loss of generality, suppose that $z = m_1$. Then cancelling z on both sides of 0.7, we obtain

$$x_1^{n-2}x_2^{s-2}\cdots x_n^{s-2} = m_2\cdots m_s M. \tag{0.8}$$

On the left-hand side of 0.8, the total degree of the x_i's is $(n-2)+(n-1)(s-2) = ns-s-n$. On the right-hand side of 0.8, each $m_\ell = x_1\cdots\hat{x}_j\cdots x_n$ for $2 \leq \ell \leq s$ and some $1 \leq j \leq n$. There are $s-1$ such m_ℓ's and since each $x_1\cdots\hat{x}_j\cdots x_n$ has degree $n-1$, we have

$$\deg(m_2\cdots m_s) = (s-1)(n-1) = ns-s-n+1,$$

that is, the total degree of the x_i's on the right-hand side of 0.8 is at least $ns-s-n+1$. Since $ns-s-n < ns-s-n+1$, we cannot have that $m_2\cdots m_s \mid x_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}$. So our assumption that $zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2} \in J_2^s$ was false in this case.

In the second case, we have $z \nmid m_\ell$ for any ℓ. Then since z appears on the left-hand side of 0.7, z must appear on the right-hand side. We have assumed that $z \nmid m_\ell$ for all ℓ so we must have $z \mid M$. Dividing by z on both sides of 0.7, we obtain

$$x_1^{n-2}x_2^{s-2}\cdots x_n^{s-2} = m_1\cdots m_s(\frac{M}{z}).$$

Since each $m_\ell = x_1\cdots\hat{x}_j\cdots x_n$ has degree $n-1$, we know that $m_1\cdots m_s$ has degree $s(n-1)$. However, we saw above that $x_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}$ has degree $ns-s-n < ns-s = s(n-1)$. So we have reached a contradiction in this case as well. We can conclude that $zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2} \notin J_2^s$ for $s > n-1$ and hence $J_2^s : (zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) \neq (1)$.

Now we claim that $(z, x_1, \ldots, x_n) \subset J_2^s : (zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2})$ for $s > n-1$. We will prove this using induction on s. We first consider the case where $s = n$. We have that $z(zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) = z(zx_1^{n-2}x_2^{n-2}\cdots x_n^{n-2})$. Since $z \in J_2$ and from (1) we know that $(zx_1^{n-2}x_2^{n-2}\cdots x_n^{n-2}) \in J_2^{n-1}$, we have that

$$z(zx_1^{n-2}x_2^{n-2}\cdots x_n^{n-2}) \in J_2^n.$$

Now consider $x_i(zx_1^{n-2}x_2^{n-2}\cdots x_n^{n-2})$ for some i. We have that $x_i(zx_1^{n-2}x_2^{n-2}\cdots x_n^{n-2}) = z(x_ix_1^{n-2}x_2^{n-2}\cdots x_n^{n-2})$. Again since $z \in J_2$ and $(x_ix_1^{n-2}x_2^{n-2}\cdots x_n^{n-2}) \in J_2^{n-1}$ by (1), it follows that

$$z(x_ix_1^{n-2}x_2^{n-2}\cdots x_n^{n-2}) \in J_2^n.$$

So $(z, x_1, \ldots, x_n) \subset J_2^s : (zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2})$ for $s = n$.

Now assume that

$$z(zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) \in J_2^s \quad \text{and} \quad x_i(zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) \in J_2^s$$

for all $1 \le i \le n$. Consider

$$z(zx_1^{n-2}x_2^{s-1}\cdots x_n^{s-1}) = z(zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2})(x_2\cdots x_n).$$

By induction we have $z(zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) \in J_2^s$ and we know by Lemma 5.3 that $(x_2\cdots x_n) = \hat{x_1}x_2\cdots x_n \in J_2$ so $z(zx_1^{n-2}x_2^{s-1}\cdots x_n^{s-1}) \in J_2^{s+1}$. Similarly, we have

$$x_i(zx_1^{n-2}x_2^{s-1}\cdots x_n^{s-1}) = x_i(zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2})(x_2\cdots x_n).$$

By induction $x_i(zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) \in J_2^s$ and since $(x_2\cdots x_n) \in J_2$, we have that

$$x_i(zx_1^{n-2}x_2^{s-1}\cdots x_n^{s-1}) \in J_2^{s+1}$$

for any i. Hence, $(z, x_1 \cdots, x_n) \subset J_2^s : (zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2})$ for $s > n-1$ and since $J_2^s : (zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) \neq (1)$, we must have that

$$J_2^s : (zx_1^{n-2}x_2^{s-2}\cdots x_n^{s-2}) = (z, x_1, \ldots, x_n).$$

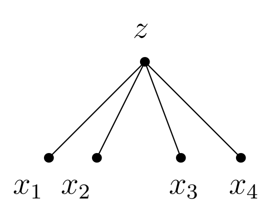
□

We demonstrate the above result in the following example.

Example 5.7. Let $K_{1,4}$ be the star on vertex set $\{z, x_1, x_2, x_3, x_4\}$. By Lemma 5.3, we know that

$$J_2 = (z, x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3).$$

The graph of $K_{1,4}$ is shown below.



$K_{1,4}$

Now by Theorem 5.6, we have that

(1) $J_2^3 : (x_1^2x_2^2x_3^2x_4^2) = (z, x_1, x_2, x_3, x_4)$, and
(2) $J_2^s : (zx_1^2x_2^{s-2}x_3^{s-2}x_4^{s-2}) = (z, x_1, x_2, x_3, x_4)$ for $s > 3$.

For example, if $s = 5$ then the annihilator of J_2^5 is given by $(zx_1^2x_2^3x_3^3x_4^3)$, i.e., $J_2^5 : (zx_1^2x_2^3x_3^3x_4^3) = (z, x_1, x_2, x_3, x_4)$. Note that in the second case, we can choose any x_i in the annihilator to have exponent $n-2$ and the rest $s-2$. This implies that not only does $J_2^5 : (zx_1^2x_2^3x_3^3x_4^3) = (z, x_1, x_2, x_3, x_4, x_5)$ but also

$$J_2^5 : (zx_1^3x_2^2x_3^3x_4^3) = J_2^5 : (zx_1^3x_2^3x_3^2x_4^3) = J_2^5 : (zx_1^3x_2^3x_3^3x_4^2) = (z, x_1, x_2, x_3, x_4).$$

This is true for any $s > 3$.

In Example 5.7, we noted that we can choose any x_i to have exponent $n-2$ and the rest $s-2$ in Theorem 5.6(2). This is true in general.

We are now ready prove our main result.

Theorem 5.8. *Let $K_{1,n}$ be a star on the vertex set $\{z, x_1, \ldots, x_n\}$ with centre z and corresponding Alexander dual J_2. Then*

(1) $(z, x_1, \ldots, x_n) \in \mathrm{Ass}(R/J_2^s)$ *for* $s \geq n-1$
(2) $(z, x_1, \ldots, x_n) \notin \mathrm{Ass}(R/J_2^s)$ *for* $s < n-1$

where $(z, x_1, \ldots, x_n) \subset k[z, x_1, \ldots, x_n]$ is a maximal ideal.

Proof. (1) This follows immediately from Theorem 5.6.
(2) We will use a proof by contradiction. Suppose that $(z, x_1, \ldots, x_n) \in \mathrm{Ass}(R/J_2^s)$ for some $s < n-1$. Then there is a monomial $m \in k[z, x_1, \ldots, x_n]$ such that $J_2^s : (m) = (z, x_1, \ldots, x_n)$. Factor out the z terms of m and write $m = z^e m'$ such that $z \nmid m'$. Then by Lemma 5.5, we have that

$$m \mid z^e(x_1 \cdots x_n)^{s-e-1}. \tag{0.9}$$

Multiplying 0.9 on both sides by $(x_1 \cdots x_n)$, we obtain

$$m(x_1 \cdots x_n) \mid z^e(x_1 \cdots x_n)^{s-e}. \tag{0.10}$$

Now since $x_1 \in (z, x_1, \ldots, x_n)$, we know that $x_1 m \in J_2^s$ and since $(\hat{x_1}x_2 \cdots x_n)$ is a generator of J_2, we have that

$$x_1 m(\hat{x_1}x_2 \cdots x_n) \;=\; m(x_1 \cdots x_n) \in J_2^{s+1}.$$

By 0.10, we know that $z^e(x_1 \cdots x_n)^{s-e}$ is a multiple of $m(x_1 \cdots x_n) \in J_2^{s+1}$ so we must have $z^e(x_1 \cdots x_n) \in J_2^{s+1}$. Hence, some generator of J_2^{s+1} divides $z^e(x_1 \cdots x_n)^{s-e}$. Every generator of J_2^{s+1} has one of the following two forms:

(*i*) z^{s+1}

(*ii*) $z^f(\hat{x_1}x_2 \cdots x_n)^{f_1} \cdots (x_1 \cdots x_{n-1}\hat{x_n})^{f_n}$, where $f \geq 0$ and $f_i \neq 0$ for some i.

We will now show that in both cases, these generators do not divide $z^e(x_1 \cdots x_n)^{s-e}$.

(*i*) Suppose that $z^{s+1} \mid z^e(x_1 \cdots x_n)^{s-e}$. Then we have $s+1 \leq e$. Which implies that $s - e \leq -1$. This contradicts the fact that $s - e \geq 0$. So $z^{s+1} \nmid z^e(x_1 \cdots x_n)^{s-e}$.

(*ii*) Now suppose that some generator of the form $z^f(\hat{x_1}x_2 \cdots x_n)^{f_1} \cdots (x_1 \cdots x_{n-1}\hat{x_n})^{f_n}$ divides $z^e(x_1 \cdots x_n)^{s-e}$. Then we must have that $0 \leq f \leq e$ and the degree of the x_i's in $z^f(\hat{x_1}x_2\cdots x_n)^{f_1}\cdots(x_1\cdots x_{n-1}\hat{x_n})^{f_n}$ is less than or equal to the degree of the x_i's in $z^e(x_1\cdots x_n)^{s-e}$. That is,

$$(n-1)(f_1+\cdots+f_n)\leq n(s-e). \tag{0.11}$$

Since $z^f(\hat{x_1}x_2\cdots x_n)^{f_1}\cdots(x_1\cdots x_{n-1}\hat{x_n})^{f_n}$ is a generator of J_2^{s+1}, we must have $f+f_1+\cdots+f_n=s+1$. Rearranging, we obtain

$$f_1+\cdots+f_n=s+1-f. \tag{0.12}$$

Substituting 0.12 into 0.11, we have

$$(n-1)(s+1-f)\leq n(s-e).$$

After expanding and cancelling terms, we get

$$f-nf+n-s-1\leq -ne.$$

Now since $0\leq f\leq e$, we have $-ne\leq -nf$. From this inequality and the fact that $f\geq 0$, we see that

$$-ne+n-s-1\leq f-nf+n-s-1\leq -ne.$$

So $-ne+n-s-1\leq -ne$. Adding $-ne$ to both sides of the inequality, we obtain

$$n-1-s\leq 0.$$

By assumption, $s<n-1$ so

$$0<n-1-s\leq 0.$$

This is impossible so our assumption that $(z,x_1,\ldots,x_n)\in \operatorname{Ass}(R/J_2^s)$ for some $s<n-1$ was false. This completes the proof. □

Theorem 5.8 tells us that not only does the maximal ideal $(z,x_1,\ldots,x_n)$ appear in the set of associated primes of J_2^s for all $s\geq n-1$ but also that this ideal appears for the first time when $s=n-1$; that is, $(z,x_1,\ldots,x_n)$ is not an associated prime for any power of J_s^2 where $s<n-1$.

In Theorem 4.6 we defined the index of stability of $\operatorname{Ass}(R/I^s)$ where $I\subset R$ is an ideal. In the following example, we illustrate how Theorem 5.8 provides us with a lower bound on the index of stability for $\operatorname{Ass}(R/J_2(K_{1,n})^s)$.

Example 5.9. Suppose that $K_{1,101}$ is a star on the vertex set $\{z,x_1,\ldots,x_{101}\}$. By Theorem 5.8 we know that

$$(z,x_1,\ldots,x_{101})\in \operatorname{Ass}(R/J_2^s),\qquad \text{for all } s\geq 100$$

and

$$(z,x_1,\ldots,x_{101})\notin \operatorname{Ass}(R/J_2^s),\qquad \text{for all } s<100.$$

Since the maximal ideal $(z,x_1,\ldots,x_{101})$ appears for the first time in $\operatorname{Ass}(R/J_2^{100})$, we can conclude that $N\geq 100$.

We end this chapter with two consequences of Theorem 5.8. In Example 5.9, we illustrated the following corollary of Theorem 5.8:

Corollary 5.10. *Let $K_{1,n}$ be a star on the vertex set $\{z, x_1, \ldots, x_n\}$ with centre z and corresponding Alexander dual J_2. Then*

$$N \geq n - 1$$

where N denotes the index of stability of $\mathrm{Ass}(R/J_2^s)$.

We will now consider how Theorem 5.8 can answer a question about hypergraphs. In [**13**], Francisco, Hà, and Van Tuyl raised the following question:

Question 5.11. *For each integer $n \geq 0$, does there exist a hypergraph $\mathcal{H}_n$ such that the stabilization of* $\mathrm{Ass}(R/J(\mathcal{H}_n)^s)$ *occurs at $N \geq (\chi(\mathcal{H}_n) - 1) + n$, where $J(\mathcal{H}_n)$ is the cover ideal corresponding to $\mathcal{H}_n$?*

Theorem 5.8 gives us an answer to this question. In order to see this, we first need to define a hypergraph. The following definition can be found in [**4**].

Definition 5.12 ((Simple) Hypergraph)**.** Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be a finite set. A *hypergraph* $\mathcal{H}$ on $\mathcal{X}$ is a family of subsets

$$\mathcal{H} = (E_1, \ldots, E_m),$$

such that

(1) $E_i \neq \emptyset$ for $i = 1, \ldots, m$, and
(2) $\bigcup_{i=1}^m E_i = \mathcal{X}$.

We call $\mathcal{X}$ the *vertex set* and $E_1, \ldots, E_m$ the *edges* of $\mathcal{H}$ respectively. If in addition, $|E_i| \geq 2$ and $E_i \not\subseteq E_j$ for $i \neq j$, then we say that $\mathcal{H}$ is *simple*.

In [**17**], Hà and Van Tuyl extended the concept of the edge ideal first defined in [**25**] to hypergraphs.

Definition 5.13 (Edge Ideal of $\mathcal{H}$)**.** Let $\mathcal{H}$ be a hypergraph on the vertex set $\mathcal{X}$ and let $\mathcal{E} = \{E_1, \ldots, E_m\}$ denote the collection of edges of $\mathcal{H}$. Then the *edge ideal* corresponding to $\mathcal{H}$ is given by

$$\mathcal{I}(\mathcal{H}) = (\{x^{E_i} \mid E_i \in E\})$$

where $x^{E_i} = \prod_{x_j \in E_i} x_j$.

We will now define a (*minimal*) vertex cover of $\mathcal{H}$ and the *cover ideal* corresponding to $\mathcal{H}$.

Definition 5.14 ((Minimal) Vertex Cover of $\mathcal{H}$)**.** Let $\mathcal{H}$ be a hypergraph on the vertex set $\mathcal{X}$ and let $\mathcal{E}$ denote the collection of edges of $\mathcal{H}$. A subset $W \subseteq \mathcal{X}$ is a *vertex cover* of $\mathcal{H}$ if $W \cap E \neq \emptyset$ for all $E \in \mathcal{E}$. A vertex cover W is *minimal* if no proper subset of W is a vertex cover of $\mathcal{H}$.

Definition 5.15 (Cover Ideal of $\mathcal{H}$)**.** Let $\mathcal{H}$ be a hypergraph on the vertex set $\mathcal{X}$ and let $\mathcal{E}$ denote the collection of edges of $\mathcal{H}$. Then we define the *cover ideal* corresponding to $\mathcal{H}$ by

$$J(\mathcal{H}) = (x_{i_1} \cdots x_{i_r} \mid W = \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a minimal vertex cover of } \mathcal{H}).$$

It can be shown that the Alexander dual of $I(\mathcal{H})$ is equal to the cover ideal corresponding to $\mathcal{H}$; that is, $I(\mathcal{H})^\vee = J(\mathcal{H})$. For our purposes, we will only need the result of Lemma 5.18. In order to describe a solution to Question 5.11, we also need to define a *d-colouring* and the *chromatic number* of a hypergraph. The following definitions can be found in [**13**].

Definition 5.16 (d-colouring of $\mathcal{H}$)**.** Let $\mathcal{H}$ be a hypergraph on the vertex set $\mathcal{X}$ and let $\mathcal{E}$ denote the collection of edges of $\mathcal{H}$. A d-colouring of $\mathcal{H}$ is a partition of $\mathcal{X} = C_1 \cup \cdots \cup C_d$ into d disjoint sets such that if $E \in \mathcal{E}$, then $E \nsubseteq C_i$ for any i. We call $C_1, \ldots, C_d$ *colour classes* of a d-colouring.

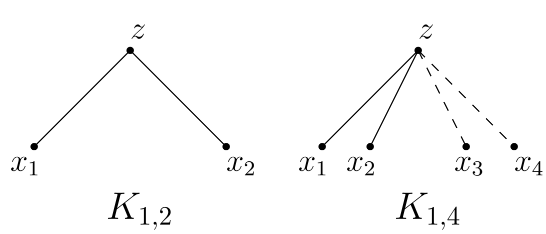
Definition 5.17 (Chromatic Number of $\mathcal{H}$)**.** The *chromatic number* of a hypergraph $\mathcal{H}$ is the minimal d such that $\mathcal{H}$ has a d-colouring. We denote the chromatic number of a hypergraph $\mathcal{H}$ by $\chi(\mathcal{H})$.

We are now ready to construct a hypergraph $\mathcal{H}_n$ that will give us a solution to Question 5.11.

Construction of $\mathcal{H}_n$:

Take a star $K_{1,n}$ on the vertex set $V = \{z, x_1, \ldots, x_n\}$. Add two vertices to V and construct a new star $K_{1,n+2}$ on the vertex set $V' = \{z, x_1, \ldots, x_n, x_{n+1}, x_{n+2}\}$.

For example, suppose that we take $K_{1,2}$ on the vertex set $\{z, x_1, x_2\}$ shown below on the left. Now we add the vertices x_3 and x_4 and construct a new star $K_{1,4}$ on the vertex set $\{z, x_1, x_2, x_3, x_4\}$ shown on the right. The dashed lines indicate the new edges that connect x_3 and x_4 to z respectively.



Now let $\mathcal{X} = V'$ denote the vertex set of our hypergraph $\mathcal{H}_n$. We define the edges of $\mathcal{H}_n$ by

$$E_i = \{x_j, z, x_k\},$$

where $1 \le j < k \le n+2$. The vertices in each E_i correspond to a path of length two in $K_{1,n+2}$.

In our example $n + 2 = 4$ so the edges of $\mathcal{H}_2$ are given by

$E_1 = \{x_1, z, x_2\}, E_2 = \{x_1, z, x_3\}, E_3 = \{x_1, z, x_4\}, E_4 = \{x_2, z, x_3\}, E_5 = \{x_2, z, x_4\}$, and $E_6 = \{x_3, z, x_4\}$.

Throughout the rest of this chapter, we will refer to the hypergraph $\mathcal{H}_n$ constructed as described above as $\mathcal{H}_n$ *corresponding to* $K_{1,n+2}$.

We now need the following two lemmas:

Lemma 5.18. *Let $\mathcal{H}_n$ be the hypergraph corresponding to $K_{1,n+2}$. Then*

$$J_2(K_{1,n+2}) = J(\mathcal{H}_n)$$

where $J_2(K_{1,n+2})$ is the Alexander dual of the path ideal $I_2(K_{1,n+2})$ and $J(\mathcal{H}_n)$ is the cover ideal corresponding to $\mathcal{H}_n$.

Proof. In Lemma 5.3, we showed that

$$J_2(K_{1,n+2}) = (z) + (x_1 \cdots \hat{x}_\ell \cdots x_{n+2} \mid \ell = 1, \ldots, n+2) \tag{0.13}$$

where $\hat{x}_\ell$ denotes the absence of vertex x_ℓ. So it is enough to show that

$$(z) + (x_1 \cdots \hat{x}_\ell \cdots x_{n+2} \mid \ell = 1, \ldots, n+2) = J(\mathcal{H}).$$

We will prove this by double inclusion. To obtain the first inclusion, we will show that each generator of 0.13 corresponds to a minimal vertex cover of $\mathcal{H}_n$. By the way we have constructed $\mathcal{H}_n$, we have $z \in E_i$ for each edge set E_i of $\mathcal{H}_n$. This implies that $E_i \cap \{z\} = \{z\} \neq \emptyset$ so $\{z\}$ is a vertex cover of $\mathcal{H}_n$. Since the only proper subset of $\{z\}$ is $\emptyset$ and $\emptyset \cap E_i = \emptyset$ for all i, we can conclude that $\{z\}$ is a minimal vertex cover of $\mathcal{H}_n$. It follows that $z \in J(\mathcal{H}_n)$. Now consider a generator of 0.13 of the form $x_1 \cdots \hat{x}_\ell \cdots x_n$. The set $W = \{x_1, \ldots, \hat{x}_\ell, \ldots, x_n\}$ intersects every E_i nontrivially except for possibly the edges containing x_ℓ. These take on two forms: $E_q = \{x_q, z, x_\ell\}$ where $q < \ell$ and $E_{q'} = \{x_\ell, z, x_{q'}\}$ where $\ell < q'$. However, $W \cap E_q = \{x_q\}$ and $W \cap E_{q'} = \{x_{q'}\}$ since the only vertices missing from W are z and x_ℓ. In both cases, W intersects the edge nontrivially and hence $W \cap E_i \neq \emptyset$ for all i. It follows that W is a vertex cover of $\mathcal{H}_n$ so $x_1 \cdots \hat{x}_\ell \cdots x_n \in J(\mathcal{H}_n)$. To show that W is a minimal vertex cover of $\mathcal{H}_n$, suppose that there is a proper subset $W' \subset W$ such that W' is a vertex cover of $\mathcal{H}_n$. Then there is $x_{\ell'}$ where $\ell' \neq \ell$ such that $x_{\ell'} \in W \setminus W'$. Consider the edge $E = \{x_\ell, z, x_{\ell'}\}$. Since $z \notin W$ and $x_\ell \notin W$, we must have that $z, x_\ell \notin W'$. We have also assumed that $x_{\ell'} \notin W'$ so $E \cap W' = \emptyset$. This contradicts the fact that W' is a vertex cover of $\mathcal{H}_n$. Hence, W is a minimal vertex cover of $\mathcal{H}_n$. Since each generator of 0.13 is in $J(\mathcal{H}_n)$, we can conclude that $J_2(K_{1,n+2}) \subseteq J(\mathcal{H}_n)$. Now let m be a monomial in $J(\mathcal{H}_n)$. We can write

$$m = x_{j_1} \cdots x_{j_r} M \tag{0.14}$$

where M is a monomial in $k[z, x_1, \ldots, x_{n+2}]$ and $W = \{x_{j_1}, \ldots, x_{j_r}\}$ is a minimal vertex cover of $\mathcal{H}_n$. We have two cases: $z \mid m$ or $z \nmid m$. If $z \mid m$, then $m \in (z)$ and we are done. If $z \nmid m$, then $z \notin W$ since if it were, we would have that $x_{j_k} = z$ for some k and $z \mid m$. Now we claim that at most one x_ℓ does not divide m. We will prove this by contradiction. Suppose that $x_\ell \nmid m$ and $x_{\ell'} \nmid m$ where $\ell < \ell'$. Then $x_\ell \notin W$, since if it were then x_ℓ would appear on the right-hand side of 0.14; that is, x_ℓ would divide m. Similarly, $x_{\ell'} \notin W$. Now consider the edge of $\mathcal{H}_n$, $E = \{x_\ell, z, x_{\ell'}\}$. Since $z, x_\ell, x_{\ell'} \notin W$, we have that $W \cap E = \emptyset$. This contradicts the assumption that W is vertex cover of $\mathcal{H}_n$. Hence, at most one x_ℓ does not divide m; that is, $x_1 \cdots \hat{x}_\ell \cdots x_{n+2} \mid m$ for some ℓ. It follows that $m \in J_2(K_{1,n+2})$ and we have the second inclusion. □

Lemma 5.19. *Let $\mathcal{H}_n$ be the hypergraph corresponding to $K_{1,n+2}$. Then $\chi(\mathcal{H}_n) = 2$.*

Proof. Let $\mathcal{H}_n$ be the hypergraph corresponding to $K_{1,n+2}$ and let $C_1 = \{x_1, \ldots, x_{n+2}\}$ and $C_2 = \{z\}$. We claim that $\mathcal{X} = C_1 \cup C_2$ is a 2-colouring of $\mathcal{H}_n$. It is easy to see that $C_1 \cap C_2 = \emptyset$ so C_1 and C_2 are disjoint. By the way we have constructed $\mathcal{H}_n$, we know that $|E_i| = 3$ for each i but $|C_2| = 1$ so $E_i \nsubseteq C_2$ for any i. Since each E_i is of the form $E_i = \{x_j, z, x_k\}$ where $1 \leq j < k \leq n+2$, we have that $z \in E_i$ for every i but $z \notin C_1$, since if it were then $C_1 \cap C_2 \neq \emptyset$. It follows from Definition 5.16 that $\mathcal{X} = C_1 \cup C_2$ is a 2-colouring of $\mathcal{H}_n$.

Now we want to show that $\chi(\mathcal{H}_n) = 2$. We will do this by contradiction. Suppose that $\mathcal{H}_n$ has a 1-colouring. Then the only possibility is $\mathcal{X} = C_1$ but $E_i \subseteq \mathcal{X}$ for all i. So $E_i \subseteq C_1$ for all i contradicting the assumption that $\mathcal{X} = C_1$ is a 1-colouring of $\mathcal{H}_n$. Hence, $\chi(\mathcal{H}_n) = 2$. □

We are finally ready to prove a corollary of Theorem 5.8 that gives a solution to Question 5.11.

Corollary 5.20. *Let $\mathcal{H}_n$ be the hypergraph corresponding to $K_{1,n+2}$. Then the stabilization of* $\operatorname{Ass}(R/J(\mathcal{H}_n)^s)$ *occurs at*

$$N \geq (\chi(\mathcal{H}_n) - 1) + n.$$

Proof. Let $\mathcal{H}_n$ be the hypergraph corresponding to $K_{1,n+2}$. By Corollary 5.10 we have that the stabilization of $\operatorname{Ass}(R/J_2(K_{1,n+2})^s)$ occurs at

$$N \geq (n+2) - 1 = (2-1) + n. \tag{0.15}$$

By Lemma 5.18, we know that $J_2(K_{1,n+2}) = J(\mathcal{H}_n)$. This implies that

$$\operatorname{Ass}(R/J_2(K_{1,n+2})^s) = \operatorname{Ass}(R/J(\mathcal{H}_n)^s). \tag{0.16}$$

Finally, we showed in Lemma 5.19 that $\chi(\mathcal{H}_n) = 2$ so combining this fact with 0.15 and 0.16, we obtain that the stabilization of $\operatorname{Ass}(R/J(\mathcal{H}_n)^s)$ occurs at

$$N \geq (2-1) + n = (\chi(\mathcal{H}_n) - 1) + n$$

as desired. □

We saw earlier how to constuct the hypergraph $\mathcal{H}_2$ corresponding to $K_{1,4}$. We will conclude this chapter with an example that illustrates the connection between Corollary 5.10 and Corollary 5.20.

Example 5.21. First consider the star $K_{1,4}$ on the vertex set $\{z, x_1, x_2, x_3, x_4\}$ with corresponding Alexander dual J_2. By Corollary 5.10 we have that

$$N \geq 4 - 1 = 3, \tag{0.17}$$

where N denotes the index of stability of $\mathrm{Ass}(R/J_2^s)$.

Now let $\mathcal{H}_2$ be the hypergraph corresponding to $K_{1,4}$ on the vertex set $\mathcal{X} = \{z, x_1, x_2, x_3, x_4\}$. By Lemma 5.19 and Corollary 5.20 we have

$$N \geq (\chi(\mathcal{H}_2) - 1) + 2 = (2 - 1) + 2 = 3,$$

which is consistent with the bound given for $\mathrm{Ass}(R/J_2(K_{1,4})^s)$ in 0.17.

CHAPTER 6

Future Work

In this chapter we discuss some conjectures based on observations made while conducting computer experiments using *Macaulay2* [**16**]. Our focus thus far has primarily been on unrooted trees. In particular, our results in Chapter 5 focused on stars $K_{1,n}$ which are complete bipartite graphs with a partition class containing a single vertex. Because of the nature of $K_{1,n}$, it was necessary that we treated the graph as an unrooted tree in order to study paths of length two.

It appears that for any rooted tree, the following conjecture holds:

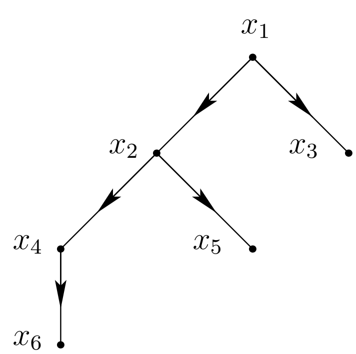
Conjecture 6.1. *Let* $\Gamma = (V_\Gamma, E_\Gamma)$ *be a rooted tree with corresponding Alexander dual* J_2*. Then*

$$\mathrm{Ass}(R/J_2) = \mathrm{Ass}(R/J_2^s)$$

for all integers $s \geq 1$*.*

Example 6.2. In Example 3.41, we saw the tree Γ_2, shown below, on the vertex set $V_{\Gamma_2} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. The Alexander dual of I_2 corresponding to Γ_2 is given by

$$J_2(\Gamma_2) = (x_2, x_1x_4, x_4x_5, x_1x_6).$$



Γ_2

Since we can write

$$J_2(\Gamma) = (x_1, x_2, x_4) \cap (x_1, x_2, x_5) \cap (x_2, x_4, x_6)$$

by the definition of J_2 (Definition 3.36), we know that

$$\mathrm{Ass}(R/J_2(\Gamma)) = \{(x_1, x_2, x_4), (x_1, x_2, x_5), (x_2, x_4, x_6)\}.$$

Using *Macaulay2*, we were able to compute $\operatorname{Ass}(R/J_2(\Gamma_2)^s)$ for $1 \leq s \leq 62$. For all such s, we found that $\operatorname{Ass}(R/J_2(\Gamma)) = \operatorname{Ass}(R/J_2(\Gamma)^s)$.

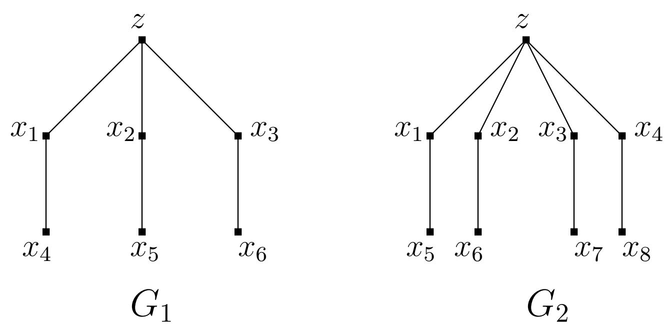
If Conjecture 6.1 holds, then the index of stability for any rooted tree is $N = 1$.

The following examples lead us to a conjecture that extends the result of Theorem 5.8 to a larger family of graphs.

Example 6.3. Let G_1 and G_2 be the unrooted trees shown below. Consider the Alexander dual J_4 of I_4 corresponding to G_1 and G_2 respectively given by

$$J_4(G_1) = (x_4, x_1, z, x_2, x_5) \cap (x_4, x_1, z, x_3, x_6) \cap (x_5, x_2, z, x_3, x_6),$$

$$J_4(G_2) = (x_5, x_1, z, x_2, x_6) \cap (x_5, x_1, z, x_3, x_7) \cap (x_5, x_1, z, x_4, x_8) \cap (x_6, x_2, z, x_3, x_7) \cap (x_6, x_2, z, x_4, x_8) \cap (x_7, x_3, z, x_4, x_8).$$



We found that

$$(z, x_1, x_2, x_3, x_4, x_5, x_6) \in \operatorname{Ass}(R/J_4(G_1)^s),$$

and

$$(z, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \operatorname{Ass}(R/J_4(G_2)^{s'})$$

for $2 \leq s \leq 10$ and $3 \leq s' \leq 5$ respectively.

Example 6.4. Let G be the unrooted tree shown below on the vertex set

$$V_G = \{z, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}.$$

The Alexander dual J_6 of I_6 is given by

$$J_6(G) = (x_7, x_4, x_1, z, x_2, x_5, x_8) \cap (x_7, x_4, x_1, z, x_3, x_6, x_9) \cap (x_8, x_5, x_2, z, x_3, x_6, x_9).$$

Then we found that

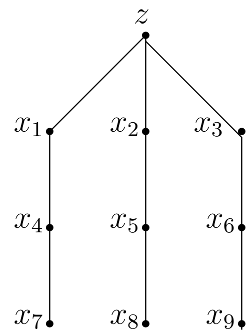
$$(z, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \operatorname{Ass}(R/J_6(G)^s)$$

for $2 \leq s \leq 6$.

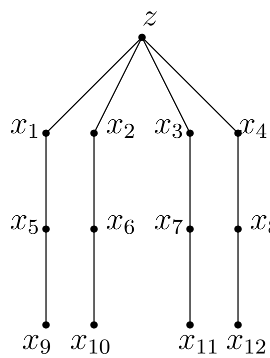
If G' is the unrooted tree shown below the graph G, then the longest even path of G' is also of length six. We found that

$$(z, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \in \operatorname{Ass}(R/J_6(G')^s)$$

for $s = 3$. We were able to compute $\operatorname{Ass}(R/J_6(G')^s)$ for $s = 1, 2, 3$.



G



G'

We observe that the ideal $(z, x_1, \ldots, x_n)$ appears in $\mathrm{Ass}(R/J_{2\ell}^s)$ corresponding to the graphs in Examples 6.3 and 6.4 for s equal to one less than the number of vertices adjacent to z. Although, due to the size of the graphs, we were only able to compute $\mathrm{Ass}(R/J_{2\ell}^s)$ for a small number of values of s, we make the following conjecture:

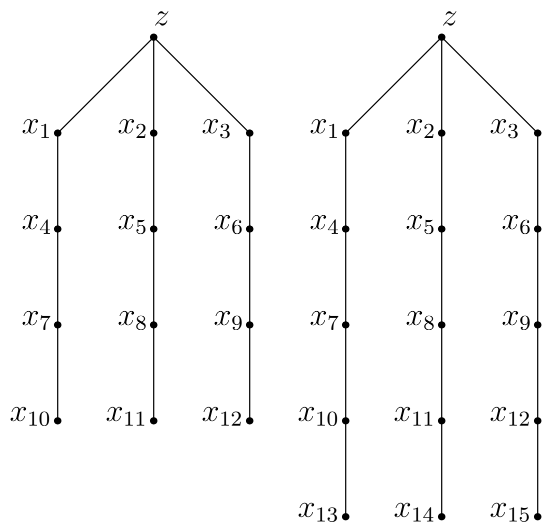
Conjecture 6.5. *Let G be an unrooted tree on the vertex set $V_G = \{z, x_1, \ldots, x_n\}$ with the edge set $E_G = \{zx_1, zx_2, \ldots, zx_k, x_1x_{k+1}, x_2x_{k+2}, \ldots, x_kx_{2k}, x_{k+1}x_{2k+1}, x_{k+2}x_{2k+2}, \ldots, x_{2k}x_{3k}, \ldots\}$. Then*

(1) $(z, x_1, \ldots, x_n) \in \mathrm{Ass}(R/J_{2\ell}^s)$ *for all* $s \geq k-1$ *and*
(2) $(z, x_1, \ldots, x_n) \notin \mathrm{Ass}(R/J_{2\ell}^s)$ *for all* $s < k-1$

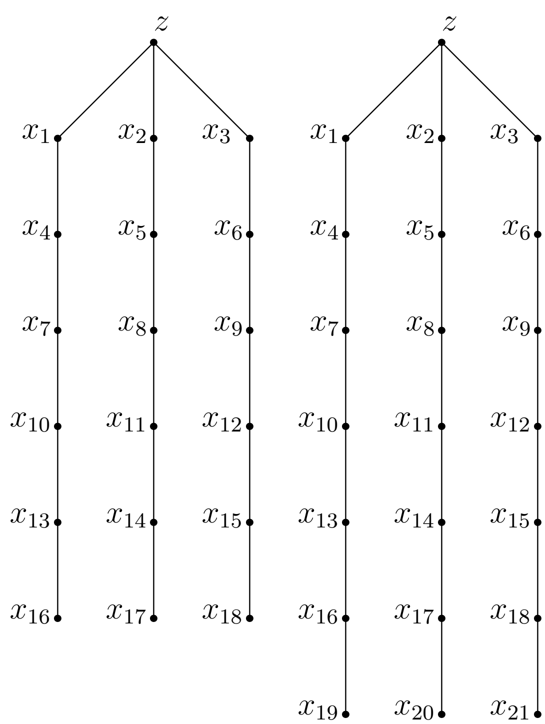
where 2ℓ is the length of the longest even path of G.

We conclude this chapter with an example showing further evidence that Conjecture 6.5 is true. Again, for each graph G we were only able to compute $\mathrm{Ass}(R/J_{2\ell}(G)^s)$ for small values of s. However with the help of a more powerful computer, we hope to test larger graphs as well as larger values of s in the future.

Example 6.6. Below we have four graphs, namely G_1, G_2, G_3, and G_4 which we treat as unrooted trees. The longest even path of G_1 is of length 8. We were able to compute $\mathrm{Ass}(R/J_8(G_1)^s)$ for $1 \leq s \leq 4$.



G_1 G_2



G_3 G_4

Moving our attention to the graphs G_2 and G_3, we can see that the longest even paths of G_2 and G_3 are of lengths ten and twelve respectively. We were able to compute both $\mathrm{Ass}(R/J_{10}(G_2)^s)$ and $\mathrm{Ass}(R/J_{12}(G_3)^s)$ for $1 \leq s \leq 3$. Finally, consider the graph G_4. The longest even path of G_4 is of length 14. We were only able to compute $\mathrm{Ass}(R/J_{14}(G_4)^s)$ for $s = 1$ and $s = 2$.

Note that in all four graphs the vertex z is adjacent to three vertices, namely x_1, x_2, and x_3. We found that

- $(z, x_1, \ldots, x_{12}) \in \mathrm{Ass}(R/J_8(G_1)^s)$ for $s = 2, 3, 4$.
- $(z, x_1, \ldots, x_{15}) \in \mathrm{Ass}(R/J_{10}(G_2)^s)$ for $s = 2, 3$.
- $(z, x_1, \ldots, x_{18}) \in \mathrm{Ass}(R/J_{12}(G_3)^s)$ for $s = 2, 3$.
- $(z, x_1, \ldots, x_{21}) \in \mathrm{Ass}(R/J_{14}(G_4)^s)$ for $s = 2$.

In all four cases, the maximal ideal $(z, x_1, \ldots, x_n) \subset k[z, x_1, \ldots, x_n]$ appeared in $\mathrm{Ass}(R/J_{2\ell}(G_i)^s)$ for $s = 2$, where $2 = 3 - 1$ is one less than the number of vertices adjacent to z. This is consistent with Conjecture 6.5.

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