Math 3T03 - Exam 1 - Solutions

Name: _____

1. (a) (10 points) Consider the set $X = \{a, b, c, d, e\}$. Determine whether the collection

$$\mathcal{T} = \{\emptyset, X, \{d, e\}, \{a, b\}\}$$

is a topology. Briefly justify your answer.

This is not a topology, since

$$\{d, e\} \cup \{a, b\} = \{a, b, d, e\}$$

is not in $\mathcal{T}$.

(b) (10 points) Consider the set $X = \{a, b, c\}$. The collection

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$

forms a topology on X (you do not need to prove this). Determine whether this topology is Hausdorff. Briefly justify your answer.

This is not Hausdorff. Every set in $\mathcal{T}$ containing c also contains a.

2. (20 points) Consider the set

$$\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$$

of positive integers, equipped with the linear order $<$ (less than). Show that the order topology is finer than the discrete topology.

Let $\mathcal{T}_{order}$ and $\mathcal{T}_{discrete}$ be the order and discrete topologies, respectively. Suppose $n \in \mathbb{Z}_+$. Then the singleton set $\{n\}$ can be written as

$$\{n\} = \begin{cases} (n-1, n+1) & \text{if } n > 1 \\ [1, 2) & \text{if } n = 1 \end{cases}$$

By definition of $\mathcal{T}_{order}$, the intervals on the right are all open in $\mathcal{T}_{order}$. It follows that $\{n\}$ is open in $\mathcal{T}_{order}$ for every $n \in \mathbb{Z}_+$.

Now let $S \in \mathcal{T}_{discrete}$ be a set that is open in the discrete topology (that is, S is any subset of $\mathbb{Z}_+$). Then

$$S = \bigcup_{n \in S} \{n\}$$

is the union of sets that are open in the order topology, and so $S \in \mathcal{T}_{order}$. Hence

$$\mathcal{T}_{order} \subseteq \mathcal{T}_{discrete},$$

as desired.

3. (20 points) Suppose $\{X_i\}_{i \in I}$ is a collection of topological spaces. Give a basis for the box topology on $\prod_{i \in I} X_i$. *Note: This is asking about the* ***box*** *topology, not the* ***product*** *topology.*

The collection

$$\left\{ \prod_{i \in I} U_i \;\middle|\; U_i \text{ is open in } X_i \right\}$$

forms a basis for the box topology.

4. (20 points) Let X be a topological space, and $A \subset X$ a subset. Suppose that for each $x \in A$ there is an open set U satisfying

$$x \in U, \qquad \text{and} \qquad U \subset A.$$

Show that A is open in X.

For each $x \in A$, let $U_x \subseteq A$ be the open set containing x asserted in the statement. Since $U_x \subset A$ for all $x \in A$, we have

$$\bigcup_{x \in A} U_x \subseteq A.$$

On the other hand, for each $x \in A$ we have $x \in U_x$. It follows that

$$A \subseteq \bigcup_{x \in A} U_x.$$

This shows $A = \bigcup_{x \in A} U_x$ is a union of open sets, and so is open.

5. (20 points) Let $f : X \to Y$ be a continuous map. Suppose $Z \subseteq Y$ is a subset with $f(X) \subseteq Z$. Give Z the subspace topology. Define a map

$$g : X \to Z$$

by $g(x) = f(x)$ for every $x \in X$. Show that g is continuous.

Let $U \subset Z$ be open in Z. Then $U = V \cap Z$ for some $V \subset Y$ that is open in Y. Since $f(X) \subset Z$, it follows that

$$f^{-1}(V) = f^{-1}(V \cap Z).$$

The definition of V shows $f^{-1}(V \cap Z) = f^{-1}(U)$. Since $f(x) = g(x)$ for all $x \in X$, we have $f^{-1}(U) = g^{-1}(U)$. Putting these together gives

$$f^{-1}(V) = g^{-1}(U). \tag{1}$$

On the other hand, we know that f is continuous, and V is open in Y, so $f^{-1}(V)$ is open in X. Then (1) implies that $g^{-1}(U)$ is open in X. Since U was arbitrary, it follows that g is continuous.