Math 3T03 - Final Exam

Name: ______________________________

Directions. This exam is open book and open note. This means that you may freely refer to the course textbook, notes from class, any material from the course webpage, and any additional notes I have given to the class. *These are the only resources at your disposal (e.g., no friends in or outside of class, no other internet sources, etc.).*

You may use, without proof, any result (theorem, lemma, etc.), assuming we have covered the result at some point in the course, and assuming the result does not directly answer the exact problem you are being asked to solve. If you do quote a result without proof, then you should provide a suitable reference (e.g., "it follows from Lemma 23.1 of Munkres that ...").

You are welcome to ask me questions. However, I may be limited in my responses.

Your final answers should be neatly written or typed. Focus on correct use of mathematical language, symbols, etc.

Submit the exam by 3 pm on Tuesday, April 19 in my office (if I am not there, then slip the exam under the door).

Problem 1. Define a relation $\sim$ on $\mathbb{R}^2$ by

$$(x, y) \sim (x', y'), \quad \text{iff} \quad \exists a, b \in \mathbb{Z}, \text{ such that } x' = x + a + b, \text{ and } y' = y + b$$

(Note that the x-components are affected differently than the y-components.)

(a) (10 points) Show that $\sim$ is an equivalence relation.

(b) (20 points) Consider the quotient space

$$\mathbb{R}^2/\sim \; = \left\{ [x] \;\middle|\; x \in \mathbb{R}^2 \right\}$$

consisting of the equivalence classes of $\sim$ on $\mathbb{R}^2$. Equip this with the quotient topology coming from the surjective function

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\sim, \qquad x \longmapsto [x].$$

Show that the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to the torus $S^1 \times S^1$. (That is, you need to define a function

$$\mathbb{R}^2/\sim \longrightarrow S^1 \times S^1,$$

prove that your function is well-defined (10 points), and also show that your function is a homeomorphism (10 points).)

**For the above problem, use the following definition of S^1:*

$$S^1 := \left\{ (s,t) \in \mathbb{R}^2 \,\middle|\, s^2 + t^2 = 1 \right\}.$$

Problem 2. Choose *four* of the following problems to complete (10 points each). You do not need to complete the remaining problems.

(a) Suppose X is a surface. Show that $X \# S^2$ is homeomorphic to X.

(b) Show that the connect sum of two real projective planes is homeomorphic to the Klein bottle:

$$\mathbb{R}P^2 \# \mathbb{R}P^2 \cong K.$$

(c) Exhibit a triangulation of the 2-sphere.

(d) Exhibit a triangulation of the Klein bottle.

Problem (e) has been removed (please choose a different problem).

(f) Prove that $\mathbb{R}^1$ with the Zariski topology is not Hausdorff.

(g) Let $\mathbb{A}^k$ be the set $\mathbb{R}^k$ equipped with the Zariski topology. Show that $\mathbb{A}^2$ is not homeomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$.

Problem 3. Consider the two-element set $\{0,1\}$ equipped with the discrete topology, and form the countably infinite product

$$X := \{0,1\}^\omega = \prod_{n \in \mathbb{Z}_+} \{0,1\}.$$

(So X consists of the infinite sequences $(x_n)_{n \in \mathbb{Z}_+}$, where, for each $k \in \mathbb{Z}_+$, the kth term x_k is either 0 or 1.) Equip X with the product topology.

Choose *four* of the following problems to complete (10 points each). You do not need to complete the remaining problem.

(a) Show that X is compact.

(b) Show that X is *totally disconnected*; that is, show that if $S \subset X$ is a subspace with at least two points, then S is not connected.

(c) Show that X is a *perfect set*; that is, show that $X = X'$, where X' is the set of limit points of X.

(d) Consider the function

$$d : X \times X \longrightarrow \mathbb{R}$$

defined by

$$d((x_n)_{n\in\mathbb{Z}_+}, (y_n)_{n\in\mathbb{Z}_+}) = 1/k,$$

where $k \in \mathbb{Z}_+$ is the smallest index where $x_k \neq y_k$. Show that d defines a metric, and show that the metric topology agrees with the product topology.

(e) Let A_0 be the closed interval $[0, 1]$. Let A_1 be the set obtained from A_0 by deleting the set $(1/3, 2/3)$; this is the 'middle third' interval. Let A_2 be the set obtained from A_1 by deleting its 'middle thirds' $(1/9, 2/9)$ and $(7/9, 8/9)$. In general, define a set A_n by

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{3+2k}{3^n} \right).$$

Finally, consider the intersection

$$C := \bigcap_{n\in\mathbb{Z}_+} A_n$$

and equip this with the subspace topology as a subset of $\mathbb{R}$.

Show that X is homeomorphic to C.

Remark 0.1. In fact, every metric space that is compact, totally disconnected, and a perfect set, is also homeomorphic to X (and hence C). (You do not need to prove this.)