Topology - Homework 3 - Partial Solutions

1. Consider the half-line $X := [0, \infty)$. In this problem we will be studying the Cartesian product X^2.

(c) Consider the map

$$\pi_1 : X^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1$$

given by projecting to the first factor. Show that π_1 is continuous when X^2 is equipped with the topology $\mathcal{T}_Q$, and $\mathbb{R}$ is equipped with the standard topology. *Hint: It suffices to show that $\pi_1^{-1}((a,b))$ is open in $(X^2, \mathcal{T}_Q)$ for each open interval $(a,b) \subset \mathbb{R}$. (If you use this hint, then you should briefly explain why this is sufficient.)*

To see this, let $U \subset \mathbb{R}$ be an open set. We want to show that $\pi_1^{-1}(U)$ is open in X^2. It suffices to consider the case where $U = (a,b)$ is an open interval. This follows by the discussion at the top of page 103, together with the fact that the open intervals form a basis for the topology on $\mathbb{R}$. Note that

$$\begin{array}{rcl} \pi_1^{-1}((a,b)) & = & \{(x_1, x_2) \mid x_1, x_2 \in [0, \infty) \quad \text{and} \quad x_1 \in (a,b)\} \\ & = & ((a,b) \cap X_1) \times X. \end{array}$$

First consider the case where $b \leq 0$. Then the above shows $\pi_1^{-1}((a,b)) = \emptyset$, which is automatically open.

Now consider the case $b > 0$. We will use the criterion for continuity given in Theorem 18.1(4). Namely fix

$$x \in \pi_1^{-1}(U)$$

(such a point exists since $b > 0$). We want to show there is some set $V \subset \pi_1^{-1}(U)$ that contains x and is open in the dictionary order topology. Using the product structure on X^2, we can write

$$x = (x_1, x_2) \in X^2$$

in terms of coordinates. Then $x \in \pi_1^{-1}(U)$ implies $x_1 \in (a,b) \cap [0, \infty)$ and $x_2 \in [0, \infty)$. We break the remaining analysis up into two cases.

Case 1: $x_2 = 0$

In this case, then define

$$V := [(0,0),(x_1,17)) .$$

This is the half open interval in X^2 relative to the dictionary order. It contains the origin $(0,0)$ on the lower end (the lowest point relative to Q), and is open on the upper end. By definition of the order topology, this is open. Moreover, it contains $x = (x_1, 0)$. This is because

$$(0,0)Q(x_1,0), \qquad (x_1,0)Q(x_1,17).$$

Case 2: $x_2 \neq 0$

In this case, it follows that $x_2 > 0$, since $x_2 \in X$. Then define

$$V := ((x_1, x_2/2),(x_1, 2x_2)) .$$

This is an open interval in the dictionary order, and so determines an open set in the dictionary order topology on X^2. Moreover, since $x_2 > 0$ we have

$$x_2/2 < x_2 < 2x_2.$$

This implies that $x \in V$.

2. Let X be a set, and suppose $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

(a) Suppose $\mathcal{T}_1$ is Hausdorff. Does it follow that $\mathcal{T}_2$ is Hausdorff?

(b) Suppose $\mathcal{T}_2$ is Hausdorff. Does it follow that $\mathcal{T}_1$ is Hausdorff?

For (a), the answer is *yes*. To see this, suppose $x_1, x_2 \in X$ are distinct points. Since $\mathcal{T}_1$ is Hausdorff, there are sets $U_1, U_2 \in \mathcal{T}_1$ with

$$x_i \in U_i, \qquad \text{and} \qquad U_1 \cap U_2 = \emptyset.$$

We are assuming $\mathcal{T}_1 \subseteq \mathcal{T}_2$, so $U_1, U_2 \in \mathcal{T}_2$. This is exactly the condition for $\mathcal{T}_2$ to be Hausdorff.

For (b), the answer is *no*. Let X be any set with more than one element. Take $\mathcal{T}_1$ to be the trivial topology, and $\mathcal{T}_2$ to be the discrete topology, so $\mathcal{T}_1 \subset \mathcal{T}_2$. Then $\mathcal{T}_1$ is not Hausdorff, but $\mathcal{T}_2$ is.