Topology - Homework 5 - Partial Solutions

Due Wednesday, February 24

0. Read Sections 21 and 23.

1. Determine which of the following functions $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are metrics. Justify your answers.

(i) $\rho(x, y) = 2|x + y|$

(ii) $\rho(x, y) = (\max(x, y) - \min(x, y))^2$

(iii) $\rho(x, y) = |x - y|^2$

(iv) $\rho(x, y) = |x - y|^{1/2}$

(i) This is note a metric since $\rho(-1, 1) = 0$, but $-1 \neq 1$.

(ii) This is note a metric since it fails the triangle inequality: $\rho(1, -1) = 4$, but $\rho(1, 0) + \rho(0, 1) = 2$.

(iii) This is exactly the same function as (ii), since $\max(x, y) - \min(x, y) = |x - y|$. In particular, it is not a metric.

(iv) This is a metric. The only tricky thing is verifying the triangle inequality. For this, note that the triangle inequality for the standard metric on $\mathbb{R}$ gives

$$|x - y| \leq |x - z| + |z - y| \leq |x - z| + |z - y| + 2|x - z|^{1/2}|z - y|^{1/2}.$$

Factoring the right-hand side gives

$$(|x - y|^{1/2})^2 \leq \left(|x - z|^{1/2} + |z - y|^{1/2}\right)^2.$$

Taking the square root of both sides verifies the triangle inequality.

2. Let $p \in [1, \infty)$, and consider the metric

$$d_p : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

on $\mathbb{R}^2$ defined by

$$d_p(\mathbf{x}, \mathbf{y}) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}.$$

(You do not need to prove this is a metric.) Consider the ball

$$B_{d_p}(\mathbf{0},1) := \left\{\mathbf{x} \in \mathbb{R}^2 \mid d_p(\mathbf{0},\mathbf{x}) < 1\right\}$$

of d_p-radius 1, centered at the origin.

(a) Sketch $B_{d_p}(\mathbf{0},1)$ for $p = 1, 3/2, 2, 4, 8$.

(b) What shape to does $B_{d_p}(\mathbf{0},1)$ approach as p approaches ∞? (You do not need to prove anything here, just draw a picture.)

This was covered in class. If you missed it, note that the ball $B_{d_p}(\mathbf{0},1)$ is the set of points $(x,y) \in \mathbb{R}^2$ such that $x^p + y^p < 1$. Just graph this set for the specified values of p. If you get stuck, first try to graph $y = \sqrt[p]{1-x^p}$, or use a graphing device (e.g., Wolfram Alpha).

3. Consider the function $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\rho(x_1,x_2) = 2\left|\int_{x_1}^{x_2} \frac{1}{1+t^2}\,dt\right|.$$

(a) Show that ρ is a metric on $\mathbb{R}$. (You may use, without proof, basic facts from calculus.)

Integration shows that the metric is $\rho(x_1,x_2) = 2|\tan^{-1}(x_2) - \tan^{-1}(x_1)|$. The only tricky thing now is verifying the triangle inequality, but this follows from the triangle inequality for $|a-b|$.

(b) What is the diameter of $\mathbb{R}$ relative to this metric? (See p. 121 for a definition of *diameter*.)

The diameter is

$$\begin{aligned}
\sup_{x_1,x_2\in\mathbb{R}} \rho(x_1,x_2) &= \sup_{x_1,x_2\in\mathbb{R}} 2|\tan^{-1}(x_2) - \tan^{-1}(x_1)| \\
&= 2\lim_{x_2\to\infty} \tan^{-1}(x_2) - 2\lim_{x_1\to-\infty} \tan^{-1}(x_1) \\
&= 2\pi.
\end{aligned}$$

(c) (Extra Credit) Suppose (X,d_X) and (Y,d_Y) are metric spaces. A function $\Phi : X \to Y$ is called an *isometry* if Φ is bijective, and if

$$d_X(z_1,z_2) = d_Y(\Phi(z_1),\Phi(z_2))$$

for all $z_1, z_2 \in X$.

Consider $\mathbb{R}$ with the metric ρ defined above. Define a metric ρ' on $S^1 \setminus \{(0,1)\}$ by

$$\rho'((\cos(\theta_1),\sin(\theta_1)),(\cos(\theta_2),\sin(\theta_2))) = |\theta_1 - \theta_2|,$$

where $\theta_i \in \mathbb{R}$ are in radians. Next, consider the function

$$\begin{array}{rcl} \Phi : \mathbb{R} & \longrightarrow & S^1 \setminus \{(0,1)\} \\ x & \longmapsto & \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right) \end{array}$$

Show that Φ is an isometry relative to the metrics ρ, ρ'.