Topology - Homework 7 - Partial Solutions

Due Wednesday, March 9

0. Read Sections 26 and 27.

1. Determine which of the following subspaces of $\mathbb{R}^2$ are connected, and which are path-connected. Carefully explain your answers.

(i) $\{(x, y) \in \mathbb{R}^2 \mid |y| = \cos(x)\}$

(ii) $\mathbb{R}^2 \backslash \mathbb{Q}^2$

(iii) $\{(x, y) \in \mathbb{R}^2 \mid y = \cos(1/x) \text{ and } x > 0\} \cup \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$

(i) We will show this is path-connected, and so it will follow that this is connected. To see that this is path-connected, let X denote the specified set, and fix $(x, y) \in X$. Then $y = \iota \cos(x)$, where $\iota = -1$, or $\iota = +1$. Define a function $\gamma : [0, 1] \to X$ by

$$\gamma(t) = (tx + (1 - t)\pi/2, \iota \cos(tx + (1 - t)\pi/2)).$$

This is a path in X from $(\pi/2, 0)$ to (x, y) (note that the point $(\pi/2, 0)$ is independent of whether $\iota = -1$ or $\iota = +1$). This shows that every point in X can be connected by a path to $(\pi/2, 0)$. Then given two points $p_0, p_1 \in X$, first find a path γ_0 from $(\pi/2, 0)$ to p_0 as above, and then find a path γ_1 from $(\pi/2, 0)$ to p_1, as above. Then

$$t \mapsto \begin{cases} \gamma_0(1 - 2t) & 0 \leq t \leq 1/2 \\ \gamma_1(2t - 1) & 1/2 < t \leq 1 \end{cases} \tag{1}$$

is a path in X from p_0 to p_1.

(ii) We will show that $\mathbb{R}^2 - \mathbb{Q}^2$ is a path-connected; it will then follow that it is connected. More generally, suppose X, Y are path-connected spaces, and $A \subset X, B \subset Y$ are any proper subsets. Then we will show that $X \times Y - (A \times B)$ is path-connected. To see this, first note that since A, B are proper subsets, there are $x' \in X - A$ and $y' \in Y - B$. Now, suppose $(x, y) \in X \times Y - (A \times B)$, is any point. There are two cases: The first is where $x \notin A$, and the second is where $y \notin A$. In the first case, we can define a path $\gamma_{(x,y)}$ in $X \times Y - (A \times B)$ from

(x', y') to (x, y) as follows: Let γ_x be a path in X from x' to x, and let γ_y be a path in Y from y' to y. Define a path $\gamma_{(x,y)}$ by setting

$$\gamma_{(x,y)}(t) := \begin{cases} (\gamma_x(2t), y') & 0 \leq t \leq 1/2 \\ (x, \gamma_y(2t-1)) & 1/2 < t \leq 1 \end{cases}$$

This is a continuous path in $X \times Y$ from (x', y') to (x, y). In fact, this is a path in $X \times Y - (A \times B)$, since $y' \notin B$ and $x \notin A$.

In the second case, define the path $\gamma_{(x,y)}$ by

$$\gamma_{(x,y)}(t) := \begin{cases} (x', \gamma_y(2t)) & 0 \leq t \leq 1/2 \\ (\gamma_x(2t-1), y) & 1/2 < t \leq 1 \end{cases}$$

This also has image in $X \times Y - (A \times B)$, since $x' \notin A$ and $y \notin B$. In either case, we have a path in $X \times Y - (A \times B)$ from (x', y') to (x, y).

In general, given elements $p_0, p_1 \in X \times Y - (A \times B)$, find a path γ_{p_0} from (x', y') to p_0, and a path γ_{p_1} from (x', y') to p_1 as above. Then these can be concatenated using the formula (1) (from part (i), above) to yield a path from p_0 to p_1.

(iii) This is connected, but not path-connected. To see it is connected, note that, just as with the usual topologists' sine curve (defined with a sine, as opposed to a cosine), the set

$$A = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; y = \cos(1/x) \text{ and } x > 0 \right\}$$

is connected, as is its closure

$$\overline{A} = A \cup (\{0\} \times [-1, 1]).$$

Now, the space in question is of the form

$$\overline{A} \cup (\{0\} \times \mathbb{R}).$$

Since $\overline{A} \cap (\{0\} \times \mathbb{R}) = \{0\} \times [-1, 1]$ is nonempty, it follows from Theorem 23.3 in the book that this space is connected.

To see it is not path-connected, just repeat the proof that the standard topologists' sine curve is not path-connected.

2. Section 23, exercise 2.

Suppose $U \subset \cup_n A_n$ is non-empty, open, and closed. To prove $\cup_n A_n$ is connected, we will show that $U = \cup_n A_n$. Since U is non-empty, there is some $p \in U$. Then $p \in A_k$ for some k. Hence $U \cap A_k$ is non-empty. This is also open and closed as a subset of A_k (with the subspace topology on A_k). Since A_k is connected, it follows that $U \cap A_k = A_k$, and so $A_k \subset U$.

Next, recall we have assumed $A_k \cap A_{k+1}$ is nonempty. Since $A_k \subset U$, this implies that $A_k \cap A_{k+1} \subset U \cap A_{k+1}$. Hence, $U \cap A_{k+1}$ is non-empty. Then, just as before, the connectedness of A_{k+1} implies $A_{k+1} \subset U$. Similarly, $A_{k-1} \cap A_k$ is non-empty, so $A_{k-1} \subset U$. Now fix any m. Then repeating this process $|m-k|$ times shows that $A_m \subset U$. Hence $\cup_n A_n \subset U$, as desired.

3. Section 23, exercise 4.

Let X be an infinite set equipped with the finite complement topology (so the only open subsets are $\emptyset$, and the sets whose complements are finite). Suppose $U \subset X$ is open, closed, and non-empty. We want to show that $U = X$. Since U is open and non-empty, it follows that its complement $X - U$ is finite. Note that, since X is infinite, this implies that U is an infinite set.

Now use the fact that U is closed. This means that its complement $X - U$ is open. Then either $X - U = \emptyset$, or $X - U$ has a finite complement. In the first case we are done since this implies $U = X$. The second case cannot happen since it implies U is a finite set, and we just saw that U is infinite.

4. Consider the three sets

$$(0,1) \subset \mathbb{R}, \qquad [0,1] \subset \mathbb{R}, \qquad S^1 \subset \mathbb{R}^2$$

each equipped with the subspace topology (so there are two intervals and a circle). Show that no pair of these three sets are homeomorphic. *Hint: Focus on two at a time. What happens when you remove a point from each?*