Topology - Homework 3

Due Tuesday, February 6

Required Problems (to be turned in)

0. Read Sections 18-20.

1. Consider the half-line $X := [0, \infty)$. In this problem we will be studying the Cartesian product X^2.

(a) Let $<_{DO}$ be the dictionary order on X^2 induced from the standard inequality on $[0, \infty)$. Does $(X^2, <_{DO})$ have a maximal (largest) element? Does it have a minimal (smallest) element?

(b) Let $\mathcal{T}_{DO}$ be the dictionary order topology on X^2 (that is, the topology generated by the order $<_{DO}$). Show that $(X^2, \mathcal{T}_{DO})$ is a Hausdorff space.

2. (a) Consider $\mathbb{R}$ with the standard topology, and $Y = [0, 1) \subset \mathbb{R}$ with the subspace topology. Determine the closure $\overline{Y}$.

(b) Consider $\mathbb{R}$ with the lower limit topology, and $Y = [0, 1) \subset \mathbb{R}$ with the subspace topology. Determine the closure $\overline{Y}$.

3. Let X be a set, and suppose $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

(a) Suppose $\mathcal{T}_1$ is Hausdorff. Does it follow that $\mathcal{T}_2$ is Hausdorff?

(b) Suppose $\mathcal{T}_2$ is Hausdorff. Does it follow that $\mathcal{T}_1$ is Hausdorff?

4. For each of the following functions, determine whether it is continuous.

(a) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \qquad x \mapsto 1/x$

(b) $f : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(c) $f : (\mathbb{R}, \mathcal{T}_\ell) \to (\mathbb{R}, \mathcal{T}_{st}), \qquad x \mapsto x$

(d) $f : (\mathbb{R}, \mathcal{T}_{st}) \to (\mathbb{R}, \mathcal{T}_\ell), \qquad x \mapsto x$

Here $\mathcal{T}_\ell$ is the lower limit topology and $\mathcal{T}_{st}$ is the standard topology.

5. Let $X := [0, \infty)$, and consider the product X^2 with the dictionary order topology (see Problem 1(b)). Equip $\mathbb{R}$ with the standard topology. Show that the projection map

$$\pi_1 : X^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1$$

is continuous. *Hint: It suffices to show that* $\pi_1^{-1}((a,b))$ *is open in* $(X^2, \mathcal{T}_{DO})$ *for each open interval* $(a,b) \subset \mathbb{R}$. *(If you use this hint, then you should briefly explain why this is sufficient.)*

Suggested Problems (not to be turned in)

A. Section 17, Exercise 6.

B. Section 17, Exercise 13.

C. Section 17, Exercise 20.

D. Section 18, Exercise 1 (if you know the ϵ-δ definition).

E. Section 18, Exercise 3.

F. Section 19, Exercise 4.