Topology - Homework 4
(Partial Solutions)

Due Thursday, March 1

Required Problems (to be turned in)

0. Read Sections 21 and 23.

1. Section 19, Exercise 3.

Here are two proofs that are essentially the same, but they use different notation. If you are having trouble with the abstract product notation, I encourage you to compare the two proofs closely, to see how the notation is used in each to the same effect.

Proof 1: Since the product topology is coarser than the box topology, it suffices to show that $\prod X_\alpha$ is Hausdorff in the product topology. Let $(x_\alpha), (y_\alpha) \in \prod X_\alpha$ be distinct points. Since these points are distinct, there is some index β so that the β-components $x_\beta \neq y_\beta$ are not equal. Since x_β, y_β are distinct elements of the Hausdorff space X_β, there are disjoint open sets $U, V \subset X_\beta$ so that $x_\beta \in U$ and $y_\beta \in V$. Define a set

$$\prod U_\alpha \subseteq \prod X_\alpha$$

by $U_\alpha = U$ if $\alpha = \beta$ and $U_\alpha = X_\alpha$ if $\alpha \neq \beta$. Then $\prod U_\alpha$ is open in the product topology (it is a basis element) and also contains (x_α) because U contains x_β. Similarly, define

$$\prod V_\alpha \subseteq \prod X_\alpha$$

by $V_\alpha = V$ if $\alpha = \beta$ and $V_\alpha = X_\alpha$ if $\alpha \neq \beta$. This is open in the product topology and contains (y_α). Moreover, the sets $\prod U_\alpha$ and $\prod V_\alpha$ are disjoint, since any element in the intersection would have β-component in $U \cap V = \emptyset$, which is impossible. This shows that $\prod X_\alpha$ is Hausdorff. $\square$

Proof 2: Since the product topology is coarser than the box topology, it suffices to show that $\prod X_\alpha$ is Hausdorff in the product topology. Let $x, y \in \prod X_\alpha$ be distinct points. Since these points are distinct, there is some index β so that the β-components $\pi_\beta(x) \neq \pi_\beta(y)$ are not equal; here $\pi_\beta : \prod X_\alpha \to X_\beta$ is the projection. Since $\pi_\beta(x), \pi_\beta(y)$ are distinct elements of the Hausdorff space X_β, there are disjoint open sets $U, V \subset X_\beta$ so that $\pi_\beta(x) \in U$ and $\pi_\beta(y) \in V$.

Since π_β is continuous in the product topology, the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are open. We also have

$$x \in \pi_\beta^{-1}(U), \qquad y \in \pi_\beta^{-1}(V).$$

Finally, these sets are disjoint, since

$$\pi_\beta^{-1}(U) \cap \pi_\beta^{-1}(V) = \pi_\beta^{-1}(U \cap V) = \pi_\beta^{-1}(\emptyset) = \emptyset.$$

This shows that $\prod X_\alpha$ is Hausdorff. $\square$

2. Let J be an indexing set, and suppose that, for each $\alpha \in J$, we are given a topological space X_α.

(a) Assume that for all but a finite number of $\alpha \in J$, the set X_α consists of exactly one point. Show that, on $\prod_{\alpha \in J} X_\alpha$, the box topology equals the product topology.

Proof 1: Since X_α consists of exactly one point for each α, it follows that $\prod X_\alpha$ consists of one point as well. Since a 1-point set has only one topology, it follows that the box topology must equal the product topology. $\square$

Proof 2: The product topology is always coarser than the box topology, so it suffices to show the opposite is true in this case. That is, it suffices to show that every box topology basis element is open in the product topology. Toward this end, let $\prod U_\alpha$ be open in the box topology. Since X_α consists of only one point for each α, it follows that U_α is either empty or all of X_α. If there is any $\beta \in J$ with $U_\beta = \emptyset$, then $\prod U_\alpha = \emptyset$, which is open in the product topology, as desired. On the other hand, if $U_\alpha \neq \emptyset$ for all $\alpha \in J$, then $\prod U_\alpha = \prod X_\alpha$, which is also open in the product topology (it is the full space). $\square$

(b) Assume that $J = \mathbb{Z}_+$, and each X_α consists of exactly two points, with X_α equipped with the discrete topology. On $\prod_{\alpha \in J} X_\alpha$, is the box topology equal to the product topology? Why?

No. To see this, for $\alpha \in \mathbb{Z}_+$ write $X_\alpha = \{x_\alpha, y_\alpha\}$, where x_α, y_α are the two points the X_α contains. Then $\{x_\alpha\}$ is open in the discrete topology on X_α, so $\prod \{x_\alpha\}$ is open in the box topology on $\prod X_\alpha$. However, $\prod \{x_\alpha\}$ is not open in the product topology, since an infinite number of its components (in fact, all of them) are not equal to X_α.

3. Section 19, Exercise 6.

Proof: First suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is a sequence that converges to some $\mathbf{x} \in \prod X_\alpha$, relative to the product (or box) topology, and fix an index β. We will show that $\pi_\beta(\mathbf{x}_1), \pi_\beta(\mathbf{x}_2), \ldots$ converges to $\pi_\beta(\mathbf{x}) \in X_\beta$. To see this, let $U \subset X_\beta$ be an open set containing $\pi_\beta(\mathbf{x})$. Since π_β is continuous in the product (and box) topology, it follows that $\pi_\beta^{-1}(U)$ is open in the product (and box) topology. We plainly have $\mathbf{x} \in \pi_\beta^{-1}(U)$, so our hypothesis on the sequence $\mathbf{x}_n$ implies that there is some $N \in \mathbb{Z}_+$ so that if $n \geq N$, then $\mathbf{x}_n \in \pi_\beta^{-1}(U)$. This implies that $\pi_\beta(\mathbf{x}_n) \in U$ for all $n \geq N$, which proves the desired convergence.

Conversely, assume that $\pi_\alpha(\mathbf{x}_n)$ converges to $\pi_\alpha(\mathbf{x})$ for all α. We will show that the $\mathbf{x}_n$ converge to $\mathbf{x}$ in the product topology (they do not necessarily converge in the box topology; see below). For this, let $\prod U_\alpha$ be a product topology basis element containing $\mathbf{x}$. Then there is a finite set of indices $\{\alpha_1, \ldots, \alpha_J\}$ so that $U_\alpha = X_\alpha$ if $\alpha \notin \{\alpha_1, \ldots, \alpha_J\}$. Focusing on this finite set of indices, for each $1 \leq j \leq J$, the convergence of $\pi_{\alpha_j}(\mathbf{x}_n)$ implies that there is some $N_j \in \mathbb{Z}_+$ so that

$$\pi_{\alpha_j}(\mathbf{x}_n) \in U_{\alpha_j}, \qquad \forall n \geq N_j.$$

Take $N = \max(N_1, \ldots, N_J)$ to be the maximum and fix $n \geq N$. For $\alpha \in \{\alpha_1, \ldots, \alpha_J\}$, we have $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$, by the definition of N. On the other hand, when $\alpha \notin \{\alpha_1, \ldots, \alpha_J\}$, we have $\pi_\alpha(\mathbf{x}_n) \in X_\alpha = U_\alpha$. This implies $\mathbf{x}_n \in \prod U_\alpha$ for all $n \geq N$, as desired.

Finally, we will show that there are setting in which the proof of the previous paragraph cannot be extended to the box topology. To see this, consider the case where $J = \mathbb{Z}_+$ and $X_\alpha = \{0, 1\}$ with the discrete topology (so the elements of $\prod X_\alpha$ are sequences of 0's and 1's). Define the sequence $\mathbf{x}_n$ by

$$\begin{array}{ccc}
\mathbf{x}_1 & = & (1,1,1,\ldots,1,1,1,\ldots) \\
\mathbf{x}_2 & = & (0,1,1,\ldots,1,1,1,\ldots) \\
\mathbf{x}_3 & = & (0,0,1,\ldots,1,1,1,\ldots) \\
\vdots & \vdots & \vdots \\
\mathbf{x}_n & = & (0,0,0,\ldots,0,1,1,\ldots) \\
\vdots & \vdots & \vdots
\end{array}$$

so the first 1 in $\mathbf{x}_n$ appears in the nth slot. Also define $\mathbf{x} = (0, 0, \ldots)$. Note that $\pi_\alpha(\mathbf{x}_n) = 0$ if $n \geq \alpha$ and $\pi_\alpha(\mathbf{x}_n) = 1$ if $n < \alpha$. In particular, for fixed α, we have

$$\lim_{n\to\infty} \pi_\alpha(\mathbf{x}_n) = \lim_{n\to\infty} 0 = 0 = \pi_\alpha(\mathbf{x}),$$

so the α-components all converge. It therefore suffices to show that the $\mathbf{x}_n$ do not converge to $\mathbf{x}$ in the box topology. To see this, note that

$$U := \{0\} \times \{0\} \times \{0\} \times \ldots$$

is open in the box topology on $\prod X_\alpha$ and contains $\mathbf{x}$. In fact, $\mathbf{x}$ is the only element contained in U. In particular, $\mathbf{x}_n \notin U$ for every $n \in \mathbb{Z}_+$, as desired. $\square$

4. Determine which of the following functions $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are metrics on $\mathbb{R}$. Justify your answers.

(a) $\rho_1(x, y) = x - y$

This is not a metric because it fails the first axiom: $\rho_1(0, 1) = -1 < 0$.

(b) $\rho_2(x,y) = \left|x - \frac{1}{2}y\right|$

This is not a metric because it fails the second axiom: $\rho_2(0,1) = \frac{1}{2} \neq 1 = \rho_1(1,0)$.

(c) $\rho_3(x,y) = |x^2 - y^2|$

This is not a metric because it fails the first axiom: $\rho_3(1,-1) = 0$, but $1 \neq -1$.

(d) $\rho_4(x,y) = |x-y|^{1/3}$

This is a metric. To see this, note that $\rho_4(x,y) \geq 0$ for all x, y. If $\rho_4(x,y) = 0$, then $|x-y| = 0$ and so $x = y$, which verifies the first axiom. For the second, we have

$$\rho_4(x,y) = |x-y|^{1/3} = |-(y-x)|^{1/3} = |y-x|^{1/3} = \rho_4(y,x).$$

Finally, to verify the triangle inequality, we have

$$\begin{aligned}
\rho_4(x,y)^3 &= |x-y| \\
&\leq |x-z| + |z-x| \\
&\leq |x-z| + 3|x-z|^{2/3}|z-y|^{1/3} + 3|x-z|^{1/3}|z-y|^{2/3} + |z-x| \\
&= (|x-z|^{1/3} + |z-y|^{1/3})^3 \\
&= (\rho_4(x,z) + \rho_4(z,y))^3
\end{aligned}$$

Taking the cube root (which is an increasing function), gives

$$\rho_4(x,y) \leq \rho_4(x,z) + \rho_4(z,y).$$

5. Consider the function $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x_1, x_2) = \left| \int_{x_1}^{x_2} \frac{1}{1+t^2}\, dt \right|$.

(a) Show that ρ is a metric on $\mathbb{R}$. (You may use, without proof, basic facts from calculus.)

(b) Sketch $B_\rho(0,r)$ for $r = \pi/6, \pi/4, \pi/3, \pi/2$ (that is, sketch 4 different pictures, one for each of these r values).

(c) (Extra Credit) Suppose (X, d_X) and (Y, d_Y) are metric spaces. A function $\Phi : X \to Y$ is called an *isometry* if Φ is bijective, and if

$$d_X(z_1, z_2) = d_Y(\Phi(z_1), \Phi(z_2))$$

for all $z_1, z_2 \in X$.

Consider $\mathbb{R}$ with the metric ρ defined above. Define a metric ρ' on $S^1 \setminus \{(0,1)\}$ by

$$\rho'((\cos(\theta_1), \sin(\theta_1)), (\cos(\theta_2), \sin(\theta_2))) = |\theta_1 - \theta_2|,$$

where $\theta_i \in \mathbb{R}$ are in radians. Next, consider the function

$$\begin{array}{rcl} \Phi : \mathbb{R} & \longrightarrow & S^1 \setminus \{(0,1)\} \\ x & \longmapsto & \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right) \end{array}$$

Show that Φ is an isometry relative to the metrics $2\rho, \rho'$.

Suggested Problems (not to be turned in)

A. Let J be an indexing set, and suppose that, for each $\alpha \in J$, we are given a topological space X_α. Assume that for all but a finite number of $\alpha \in J$, the set X_α has the trivial topology. Show that, on $\prod_{\alpha \in J} X_\alpha$, the box topology equals the product topology.

B. Section 19, Exercise 7.

C. Section 20, Exercise 2.

D. Let $p \in [1, \infty)$, and consider the metric

$$d_p : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

on $\mathbb{R}^2$ defined by

$$d_p(\mathbf{x}, \mathbf{y}) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p \right)^{1/p}.$$

(You do not need to prove this is a metric.) Consider the ball

$$B_{d_p}(\mathbf{0}, 1) := \left\{ \mathbf{x} \in \mathbb{R}^2 \mid d_p(\mathbf{0}, \mathbf{x}) < 1 \right\}$$

of d_p-radius 1, centered at the origin.

(a) Sketch $B_{d_p}(\mathbf{0}, 1)$ for $p = 1, 3/2, 2, 4, 8$.

(b) To what shape does $B_{d_p}(\mathbf{0}, 1)$ approach as p approaches ∞? (You do not need to prove anything here, just draw a picture.)