Topology - Homework 4

Due Thursday, March 1
(Note that this due date has been changed from the original posting)

Required Problems (to be turned in)

0. Read Sections 21 and 23.

1. Section 19, Exercise 3.

2. Let J be an indexing set, and suppose that, for each $\alpha \in J$, we are given a topological space X_α.

(a) Assume that for all but a finite number of $\alpha \in J$, the set X_α consists of exactly one point. Show that, on $\prod_{\alpha \in J} X_\alpha$, the box topology equals the product topology.

(b) Assume that $J = \mathbb{Z}_+$, and each X_α consists of exactly two points, with X_α equipped with the discrete topology. On $\prod_{\alpha \in J} X_\alpha$, is the box topology equal to the product topology? Why?

3. Section 19, Exercise 6.

4. Determine which of the following functions $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are metrics on $\mathbb{R}$. Justify your answers.

(a) $\rho_1(x, y) = x - y$

(b) $\rho_2(x, y) = \left| x - \frac{1}{2}y \right|$

(c) $\rho_3(x, y) = |x^2 - y^2|$

(d) $\rho_4(x, y) = |x - y|^{1/3}$

5. Consider the function $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x_1, x_2) = \left| \int_{x_1}^{x_2} \frac{1}{1+t^2}\, dt \right|$.

(a) Show that ρ is a metric on $\mathbb{R}$. (You may use, without proof, basic facts from calculus.)

(b) Sketch $B_\rho(0, r)$ for $r = \pi/6, \pi/4, \pi/3, \pi/2$ (that is, sketch 4 different pictures, one for each of these r values).

(c) (Extra Credit) Suppose (X, d_X) and (Y, d_Y) are metric spaces. A function $\Phi : X \to Y$ is called an *isometry* if Φ is bijective, and if

$$d_X(z_1, z_2) = d_Y(\Phi(z_1), \Phi(z_2))$$

for all $z_1, z_2 \in X$.

Consider $\mathbb{R}$ with the metric ρ defined above. Define a metric ρ' on $S^1 \setminus \{(0,1)\}$ by

$$\rho'((\cos(\theta_1), \sin(\theta_1)), (\cos(\theta_2), \sin(\theta_2))) = |\theta_1 - \theta_2|,$$

where $\theta_i \in \mathbb{R}$ are in radians. Next, consider the function

$$\begin{array}{rcl} \Phi : \mathbb{R} & \longrightarrow & S^1 \setminus \{(0,1)\} \\ x & \longmapsto & \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right) \end{array}$$

Show that Φ is an isometry relative to the metrics $2\rho, \rho'$. (The original version had a typo. The correct version is as stated here (with at 2 in front of ρ).)

Suggested Problems (not to be turned in)

A. Let J be an indexing set, and suppose that, for each $\alpha \in J$, we are given a topological space X_α. Assume that for all but a finite number of $\alpha \in J$, the set X_α has the trivial topology. Show that, on $\prod_{\alpha \in J} X_\alpha$, the box topology equals the product topology.

B. Section 19, Exercise 7.

C. Section 20, Exercise 2.

D. Let $p \in [1, \infty)$, and consider the metric

$$d_p : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

on $\mathbb{R}^2$ defined by

$$d_p(\mathbf{x}, \mathbf{y}) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}.$$

(You do not need to prove this is a metric.) Consider the ball

$$B_{d_p}(\mathbf{0}, 1) := \left\{ \mathbf{x} \in \mathbb{R}^2 \mid d_p(\mathbf{0}, \mathbf{x}) < 1 \right\}$$

of d_p-radius 1, centered at the origin.

(a) Sketch $B_{d_p}(\mathbf{0}, 1)$ for $p = 1, 3/2, 2, 4, 8$.

(b) To what shape does $B_{d_p}(\mathbf{0}, 1)$ approach as p approaches ∞? (You do not need to prove anything here, just draw a picture.)