Topology - Homework 6
(Partial Solutions)

Due Thursday, March 29

Required Problems (to be turned in)

0. Read Sections 26 and 27.

1. Consider the three sets

$$(0,1) \subset \mathbb{R}, \qquad [0,1] \subset \mathbb{R}, \qquad S^1 \subset \mathbb{R}^2$$

each equipped with the subspace topology (so there are two intervals and a circle). Show that no pair of these three sets are homeomorphic. *Hint: Focus on two at a time. What happens when you remove a point from each?*

Proof 1: First we give a proof that only uses connectedness.

That $(0,1)$ is not homeomorphic to S^1 follows because removing any point from $(0,1)$ results in a space that is disconnected, while removing any point from S^1 results in a space that is connected. In more detail, suppose $f : S^1 \to (0,1)$ is a homeomorphism, and fix any point $p \in S^1$. Then the restriction

$$f| : S^1 - \{p\} \to (0,1) - \{f(p)\}$$

is also a homeomorphism. The space $S^1 - \{p\}$ is connected (it is homeomorphic to an interval), so this implies that $(0,1) - \{f(p)\}$ is also connected. Of course, this is a contradiction, since $(0,1) - \{f(p)\}$ is not connected; indeed, the sets

$$(0, f(p)), \qquad (f(p), 1)$$

form a separation of $(0,1) - \{f(p)\}$.

To see that $[0,1]$ is not homeomorphic to $(0,1)$ using connectedness, note that if you remove 0 from $[0,1]$, then you obtain $(0,1]$, which is connected. However, no matter which point you remove from $(0,1)$, you always obtain a disconnected space. This can be formalized precisely using a contradiction as with S^1 above (as well as the next situation below).

To see that $[0,1]$ is not homeomorphic to S^1, note that if you remove the two points $0, 1$ from $[0,1]$, then you obtain $(0,1)$, which is connected. However, if you remove any two (distinct) points from S^1, then you obtain something that is disconnected. Once again, this can be formulated precisely by a contradiction argument: Assume there is a homeomorphism $f : [0,1] \to S^1$. Then the restriction $f| : (0,1) \to S^1 - \{f(0), f(1)\}$ is a homeomorphism. Since $(0,1)$ is connected, this implies that $S^1 - \{f(0), f(1)\}$ is connected. However, since $f(0) \neq f(1)$ (f is injective), it follows that $S^1 - \{f(0), f(1)\}$ is disconnected (it is homeomorphic to the union of two disjoint open intervals); this is a contradiction. $\square$

(Partial) Proof 2: Here is a (partial) proof that uses compactness (it is only a partial proof, because it does not address whether $[0,1]$ and S^1 are homeomorphic): It was discussed in class that the set $(0,1)$ is non-compact, while the sets $[0,1]$ and S^1 are compact. Therefore $(0,1)$ and $[0,1]$ are not homeomorphic, and $(0,1)$ and $[0,1]$ are not homeomorphic.

$\square$

2. Determine which of the following subsets of $\mathbb{R}^2$ are compact.

(i) $\{(x,y) \in \mathbb{R}^2 \mid (x/3)^2 + (y/5)^2 = 1\}$

Proof: This is compact. To see this, recall that for $\mathbb{R}^n$, a subset is compact if and only if it is closed and bounded. This is the zero set of a continuous function, and so it is closed (see problem 2 below for more details of this type of argument). It is also bounded, since it is contained in the ball of radius 5 centered at the origin. $\square$

(ii) $\{(x,y) \in \mathbb{R}^2 \mid (x/3)^2 - (y/5)^2 = 1\}$

This is not compact. To see this, recall that for $\mathbb{R}^n$, a subset is compact if and only if it is sequentially compact. Consider the sequence of points $(n, 5\sqrt{(n/3)^2 - 1})$ for $n \in \mathbb{Z}_+$. This lies in the above set and has not convergent subsequence (the x-coordinates diverge to $+\infty$). $\square$

(iii) $\{(x,y) \in \mathbb{R}^2 \,|x,y \in \mathbb{Q}, \text{ and } |x| + |y| \leq 1\}$

This is not compact since it is not closed: The point $(1/\pi, 1/\pi)$ is a limit point of this set that is not contained in this set. $\square$

3. Suppose X is a Hausdorff space, and let $A, B \subset X$ be disjoint compact subspaces. Show that there exist disjoint open sets U and V such that

$$A \subset U, \quad \text{and} \quad B \subset V.$$

4. Let X be a topological space, $E \subset X$ a compact subset, and $F \subset X$ a closed subset. Prove that $E \cap F$ is compact. (Note: In this question it is *not* assumed that X is Hausdorff.)

5. Show that *sequential compactness* (see Munkres p.179) is a topological property. That is, show that if X is sequentially compact and $h : X \to Y$ is a homeomorphism, then Y is sequentially compact.

Suggested Problems (not to be turned in)

A. Define the *n-sphere* to be the subspace

$$S^n := \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k=0}^{n} x_k^2 = 1 \right\}.$$

Show that S^n is compact.

Proof: Since S^n is a subset of $\mathbb{R}^{n+1}$, it suffices to show that S^n is closed and bounded. It is obviously bounded since it is contained in the ball of radius 2 centered at the origin. To see that it is closed, consider the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$f(x_0, \ldots, x_n) = \sum_{k=0}^{n} x_k^2.$$

This is a continuous function, so the inverse image of the closed set $\{1\}$ is closed. This shows S^n is closed since

$$f^{-1}(1) = S^n.$$

$\square$

B. Suppose X is a topological space, and

$$C_1, \ldots, C_n \subset X$$

are compact subspaces. Show that the union $\cup_{j=1}^{n} C_j$ is compact.