Topology - Homework 7
(Partial Solutions)

Due Friday, April 6

Required Problems (to be turned in)

0. The material for the rest of the term is mostly taken from Sections 22, 36, 51, 52, 54, and 74-78 (you are not responsible for all material in these sections, only those items discussed in class).

1. Classify all compact discrete spaces, up to homeomorphism. (A topological space $(X, \mathcal{T})$ is a *discrete space* if $\mathcal{T}$ is the discrete topology on X.)

The compact discrete spaces, up to homeomorphism, are exactly the finite sets. More specifically, we will prove the following.

Theorem 0.1. *Let X be a set with the discrete topology.*

(A) X is compact if and only if it is homeomorphic to a discrete space on the following list:

$$\emptyset, \quad \{1\}, \quad \{1,2\}, \quad \{1,2,3\}, \quad \{1,2,3,4\}, \quad \ldots. \tag{1}$$

(B) If X, Y are a pair of distinct spaces on the list (1) then X and Y are not homeomorphic.

(Item (A) says that the list (1) is complete. Item (B) says that the list (1) has no redundancies.)

Proof. First we show that everything on the list (1) is compact in the discrete topology. Let S be a set on this list. If S is empty, then it is automatically compact. We may therefore assume that $S = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Suppose $\mathcal{A}$ is an open cover of S. Then for each $k \in S$, there is some $A_k \in \mathcal{A}$ with $k \in A_k$. Then the collection $\{A_1, \ldots, A_k\}$ is a finite subcover.

Conversely, suppose that X is a compact discrete space. Consider the cover $\mathcal{A}$ given by $\{\{x\} \mid x \in X\}$. This is an open cover, since X has the discrete topology. Moreover, the only subcover of $\mathcal{A}$ is $\mathcal{A}$ itself, since removing any point $\{x\}$ would result in something that doesn't cover $x \in X$.

Since X is compact, the open cover $\mathcal{A}$ has a finite subcover $\mathcal{A}'$. But we just saw that this forces $\mathcal{A} = \mathcal{A}'$. This implies that $\mathcal{A}$ is finite, so we can write

$$\mathcal{A} = \{\{x_1\}, \ldots, \{x_n\}\}.$$

There is a bijective correspondence between X and $\mathcal{A}$; this is simply given by sending x to $\{x\}$. It follows that $X = \{x_1, \ldots, x_n\}$ is also finite. Define a function

$$\phi : \{1, \ldots, n\} \to X$$

by $\phi(n) = x_n$. It follows from the above that this is a bijection. Both ϕ and ϕ^{-1} are continuous because any map from a discrete space is continuous. It follows that ϕ is a homeomorphism, so X is homeomorphic to an element of the list. This finishes the proof of (A).

To prove (B), suppose X, Y are distinct elements of the list (1). Then X and Y have different cardinalities. In particular, they cannot be homeomorphic, because homeomorphisms are bijections, and there are no bijections between sets of different cardinalities. $\square$

2. Suppose M is a manifold and $\phi : M \to N$ is a homeomorphism. Show that N is a manifold.

Proof: First we show that N is Hausdorff. For this, let $x_1, x_2 \in N$ be distinct points. Then $\phi^{-1}(x_1), \phi^{-1}(x_2) \in M$ are distinct. Since M is Hausdorff, there are disjoint open set $U, V \subset M$ containing $\phi^{-1}(x_1), \phi^{-1}(x_2)$. Since ϕ^{-1} is continuous, the sets $\phi(U), \phi(V)$ are open in N. These contain x_1, x_2, respectively and are disjoint because ϕ is injective. This verifies the first manifold axiom.

To see that N has a countable basis for its topology, use the fact that M has a countable basis $\mathcal{B}_M = \{B_k\}_{k \in \mathbb{Z}_+}$ for its topology. Then $\mathcal{B}_N := \{\phi(B_k)\}_{k \in \mathbb{Z}_+}$ is a collection of open sets in N (because ϕ^{-1} is continuous). We need to show that $\mathcal{B}_N$ is a basis and that the topology it generates equals the topology on N. First we show that it forms a basis: Let $x \in N$. Since $\mathcal{B}_M$ forms a basis on M, there is some $B_k \in \mathcal{B}_M$ containing $\phi^{-1}(x)$, so $x \in \phi(B_k)$. This verifies the first basis axiom for $\mathcal{B}_N$. As for the second basis axiom, suppose $x \in \phi(B_k) \cap \phi(B_\ell)$ for some $\phi(B_k), \phi(B_\ell) \in \mathcal{B}_N$. Then $\phi^{-1}(x) \in B_k \cap B_\ell$, so there is some $B_j \in \mathcal{B}_M$ with $\phi^{-1}(x) \in B_j \subseteq B_k \cap B_\ell$. Then $x \in \phi(B_j) \subseteq \phi(B_k) \cap \phi(B_\ell)$. This verifies the second basis axiom.

Now we need to show that the topology generated by $\mathcal{B}_N$ equals the topology on N. Since ϕ^{-1} is continuous, each $B_k \in \mathcal{B}_N$ is open in N, so the topology generated by $\mathcal{B}_N$ is contained in the topology on N. To show the reverse inclusion, let U be open in N and $x \in U$. Since ϕ is continuous, $\phi^{-1}(U)$ is open in M. Since $\mathcal{B}_M$ generates the topology on M, there is some $B_k \in \mathcal{B}_M$ with $\phi^{-1}(x) \in B_k \subseteq \phi^{-1}(U)$. It follows that $x \in \phi(B_k) \subseteq U$, so U is open in the topology generated by $\mathcal{B}_N$. This finishes the verification of the second manifold axiom.

Finally, we need to verify the third manifold axiom, which states that N is locally homeomorphic to an open subset of $\mathbb{R}^n$. For this, let $x \in N$. Since M is a manifold, there are open sets $U \subset M$ and $V \subset \mathbb{R}^n$ and a homeomorphism $\psi : V \to U$ so that $\phi^{-1}(x) \in U$. Then $\phi(U) \subseteq N$ is an open set containing x

and $\phi \circ \psi : V \to \phi(U)$ is a homeomorphism between $\phi(U)$ and the open set $V \subseteq \mathbb{R}^n$. □

3. A connected surface S is called *prime* if $S \ncong S^2$ and the following holds: If $S \cong M_1 \# M_2$ for surfaces M_1, M_2, then $M_1 \cong S^2$ or $M_2 \cong S^2$. Find all compact, connected, prime surfaces, and briefly justify your answer.

Answer: The only ones are T^2 and $\mathbb{R}P^2$. This is all you need to receive credit for this problem, but here is an ellaboration.

Theorem 0.2. *Suppose X is a compact, connected surface. Then X is prime if and only if it is homeomorphic to T^2 or $\mathbb{R}P^2$.*

Proof. Suppose X is a compact, connected surface that is not homeomorphic to T^2 or $\mathbb{R}P^2$. Then by the classification of surfaces, S is homeomorphic to one of the following: S^2, a connect sum of at least two copies of T^2, or a connect sum of at least 2 copies of $\mathbb{R}P^2$. All of these are not prime by definition, so S is not prime.

Now we prove the converse. That is, we will show that T^2 and $\mathbb{R}P^2$ are prime. We will argue this for T^2; the argument for $\mathbb{R}P^2$ is similar. If T^2 were not prime, then since T^2 is not homeomorphic to S^2, it would follow that we could write $T^2 \cong M_1 \# M_2$ for some connected, compact surfaces M_1, M_2, with M_1 and M_2 not homeomorphic to S^2. Then by the classification of surfaces, we could write M_1 and M_2 as a connect sum of at least one copy of T^2 or at least one copy of $\mathbb{R}P^2$. Using the relation $\mathbb{R}P^2 \# T^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, it would then follow that T^2 is homeomorphic to at least two copies of T^2, or at least two copies of $\mathbb{R}P^2$. Both of these are not possible (e.g., the fundamental group of T^2 differs from both of these options), so we have a contradiction. This implies that T^2 is prime. □

Extra Credit: Suppose S is a compact, connected surface. Show that there are a finite number of compact, connected, prime surfaces $M_1, M_2, \ldots, M_k$ so that $S \cong M_1 \# M_2 \# \ldots \# M_k$.

Suggested Problems (not to be turned in)

A. Section 26, Exercise 4.