Topology - Lecture 2

Friday, January 5

1 A few fun facts

Last time, I gave some contextualization, history of, and motivation for the development of topology. However, I failed to give an interesting application of topology. Here is one:

Fun Fact 1. *At any point in time, there are two points on the equator of the earth that have the same temperature, and these points are antipodal*[1]*.*

I find this fascinating, largely because my real world experience gives me no intuition for why this is true, of even that it is true. However, my experience with topology gives me some very strong intuition why this is true.

Here is a related fact:

Fun Fact 2. *At any point in time, there are two antipodal points on the surface of the earth that have the same temperature and the same barometric pressure.*

These are both special cases of the following famous theorem in topology:

Theorem (Borsuk-Ulam, 1930)**.** *Let S^n be the n-dimensional sphere, and $f : S^n \to \mathbb{R}^n$ a continuous function. Then there is a point $x \in S^n$ so that $f(x) = f(-x)$.*

The result in Fun Fact 1 follows from Theorem 1 by considering $n = 1$, with $f : S^1 \to \mathbb{R}$ the function recording the tempurature at each point (the equator is S^1, which is a circle). The result in Fun Fact 2 is the $n = 2$ case of Theorem 1, with the components of f being given by temperature and barometric pressure (in the $n = 2$ case, the function f takes values in $\mathbb{R}^2$, and so has two components).

A full proof of the Bosuk-Ulam Theorem is beyond the scope of this class, but it is discussed further in the Section 57 of the textbook. Now let's get to the main content of the course.

Remark 1. You are responsible for the material that follows this point (and not responsible for the material that has come before this point).

[1] Two points x, y on a sphere are *antipodal* if the line connecting them passes through the center of the sphere; equivalently, if $x = -y$.

2 The axioms of topology

The most important definition of this class is the following:

Definition 1 (Topological Axioms)**.** *Let X be a set. A* **topology on** *X is a collection $\mathcal{T}$ of subsets of X satisfying the following axioms:*

- *(Non-Triviality Axioms) The collection $\mathcal{T}$ contains the empty set and X:*

$$\emptyset \in \mathcal{T}, X \in \mathcal{T}.$$

- *(Arbitrary Unions Axiom) The union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$:*

$$\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T} \Longrightarrow \cup_{\alpha \in A} U_\alpha \in \mathcal{T}.$$

- *(Finite Intersections Axiom) The intersection of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$:*

$$\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T} \text{ and } A \text{ is finite} \Longrightarrow \cap_{\alpha \in A} U_\alpha \in \mathcal{T}.$$

The pair $(X, \mathcal{T})$ is called a **topological space***, and the elements of $\mathcal{T}$ are the* **open sets***.*

The last two axioms can be expressed more concisely by saying "$\mathcal{T}$ is closed under arbitrary unions and finite intersections".

Referring back to Lecture 1, think of X as being things like $\mathbb{R}^n$, or various spaces of functions, or a Klein bottle. The topology $\mathcal{T}$ encodes the notion of convergence (or limits) that we want to impose on the the set X. Exactly what I mean by this should become clear over the next few weeks as we study various examples and show how notions of convergence and continuity can be cast in the framework of a topological space. First, however, let's look at some relatively simple examples, to get a sense for how the axioms work.

Remark 2. The topological axioms, and the examples that follow, may involve some notation with which you are not familiar (e.g., the abstract index notation appearing in the union and intersection axioms). This material is covered in more detail in Section 3 below, as well as in §1 of the textbook. I have chosen to give the axioms and examples first by way of motivating the set theory discussion necessary to understand them fully. One could equally well cover the set theory first and then cover the axioms and examples. If you prefer this approach, feel free to read Section 3 first, and return to Section 2.1 afterward.

2.1 Examples

The following examples show how to prove, directly from the definition, that something is (or is not) a topology. Throughout the discussion, we develop some tools that may help you write your own proofs.

Example 1. *Consider the set $X = \{a, b, c\}$, and the collection*

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}\} .$$

Is $\mathcal{T}_1$ a topology on X?

Yes. *To prove this, note that $\mathcal{T}_1$ clearly contains $\emptyset$ and X, so it suffices to show that $\mathcal{T}_1$ is closed under arbitrary unions and finite intersections.*

To check the arbitrary unions axiom, suppose $\{U_\alpha\}_{\alpha \in A}$ is a subcollection of $\mathcal{T}_1$. We want to show that the union

$$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_1$$

is contained in $\mathcal{T}_1$. We will consider several special cases that will be useful in simplifyin the discussion.

First consider the case where $U_\beta = X$ for some $\beta \in A$. In this case, we have

$$\bigcup_{\alpha \in A} U_\alpha = U_\beta \cup \Big(\bigcup_{\alpha \in A - \{\beta\}} U_\alpha \Big) = X \cup \Big(\bigcup_{\alpha \in A - \{\beta\}} U_\alpha \Big) = X,$$

where the last equality holds because $\bigcup_{\alpha \in A - \{\beta\}} U_\alpha$ is a subset of X (why?). Since X is in $\mathcal{T}_1$, we would be done in this case.

Next, consider the case where $U_\beta = \emptyset$ for some $\beta \in A$. In this case,

$$\bigcup_{\alpha \in A} U_\alpha = U_\beta \cup \Big(\bigcup_{\alpha \in A - \{\beta\}} U_\alpha \Big) = \emptyset \cup \Big(\bigcup_{\alpha \in A - \{\beta\}} U_\alpha \Big) = \bigcup_{\alpha \in A - \{\beta\}} U_\alpha .$$

This means that any time an empty set appears, it can be omitted from the union.

Finally, suppose there are $\beta, \gamma \in A$ with $U_\beta = U_\gamma$, but $\beta \neq \gamma$. Then

$$\bigcup_{\alpha \in A} U_\alpha = U_\beta \cup U_\gamma \cup \bigcup_{\alpha \in A - \{\beta, \gamma\}} U_\alpha = U_\beta \cup \bigcup_{\alpha \in A - \{\beta, \gamma\}} U_\alpha = \bigcup_{\alpha \in A - \{\gamma\}} U_\alpha .$$

This means that we can assume all elements of $\{U_\alpha\}_{\alpha \in A}$ are distinct (i.e., we can assume the map $\alpha \mapsto U_\alpha$ is injective).

Remark 3. The strategy employed here works much more generally; i.e., for any X and collection $\mathcal{T}$ of subsets. More specifically, when checking whether a collection $\mathcal{T}$ is closed under arbitrary unions, it suffices to assume that none of the sets in the union are equal to $\emptyset$, none of them are equal to X, and they are all distinct. The driving force here are the following set theoretic identities

$$U \cup \emptyset = U, \qquad U \cup X = X \text{ for } U \subseteq X, \qquad U \cup U = U.$$

(Where exactly are these identities used in the example above?)

Similarly, when checking whether a collection $\mathcal{T}$ is closed under finite intersections, it suffices to assume that none of the sets in the intersection are equal to $\varnothing$, none of them are equal to X, and all of them are distinct. (Why? *Hint: Use the identities*

$$U \cap \varnothing = \varnothing, \qquad U \cap X = U \text{ for } U \subseteq X, \qquad U \cup U = U.)$$

Returning to the proof of the arbitrary unions axiom, it follows from the above remark that we may assume each $U_\alpha \neq X$, $U_\alpha \neq X$. Since $\mathcal{T}_1$ only contains 3 elements and each $U_\alpha \in \mathcal{T}_1$ is not equal to two of them, we must have $U_\alpha = \{a\}$ is the third one; this is true for all $\alpha \in A$. We also saw that we may assume that all U_α are distinct. Since all U_α equal the same thing (namely $\{a\}$), the assumptions imply that there is only one element in $\{U_\alpha\}_{\alpha \in A}$, and this element if $\{a\}$:

$$\{U_\alpha\}_{\alpha \in A} = \{\{a\}\}.$$

Hence

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} \{a\} = \{a\}.$$

Since $\{a\} \in \mathcal{T}_1$, this shows that $\mathcal{T}_1$ is closed under arbitrary unions.

Moving on to the finite intersections axiom, suppose $\{U_\alpha\}_{\alpha \in A}$ is a finite subcollection of elements of $\mathcal{T}_1$ (so the indexing set A is finite; however, it turns out that this does not matter for what occurs below). By Remark 3, and the fact that $\mathcal{T}_1$ only contains three elements, we may assume the U_α are distinct and none of them are $\varnothing$ or X. It follows that $\{U_\alpha\}_{\alpha \in A} = \{\{a\}\}$ contains only one element. Consequently, the intersection

$$\bigcap_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} \{a\} = \{a\} \in \mathcal{T}_1$$

is in $\mathcal{T}_1$, so this finishes the proof. □

This example shows that it can be rather difficult to prove something is a topology. One reason for this is that the set-theoretic notation is a bit cumbersome if you are not accustomed to it; we will talk more about this shortly. Even once you get used to the notation, it can still take time to prove something is a topology. That said, many of the theorems we prove over the next few weeks are there precisely to make this work easier for you, and you should view these as tools at your disposal. Remark 3 is another such tool.

It is often easier to show something is *not* a topology, as the next example shows.

Example 2. *Consider the same set $X = \{a, b, c\}$ as in the previous example, but with*

$$\mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b\}\}.$$

Is $\mathcal{T}_2$ a topology on X?

No. *To see this, simply notice that $\{a\}, \{b\} \in \mathcal{T}_2$ are in $\mathcal{T}_2$, but the union $\{a\} \cup \{b\} = \{a, b\}$ is not in $\mathcal{T}_2$.* □

By removing $\{b\}$ from $\mathcal{T}_2$, we are left with $\mathcal{T}_1$, which we already saw is a topology. Similarly, the next example shows that, by adding elements to $\mathcal{T}_2$, we can also obtain a topology.

Example 3. *Consider the same set $X = \{a, b, c\}$ as in the previous example, but with*

$$\mathcal{T}_3 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}.$$

Is $\mathcal{T}_3$ a topology on X?

Yes. *To prove this, first note that $\emptyset, X \in \mathcal{T}_3$, so the first two axioms are satisfied. To see that $\mathcal{T}_3$ is closed under arbitrary unions and finite intersections, let $\{U_\alpha\}_{\alpha \in A}$ be a subcollection of elements in $\mathcal{T}_3$. By Remark 3, we may assume that none of the elements of $\{U_\alpha\}_{\alpha \in A}$ are equal to $\emptyset$ or X. This means that each U_α is one of $\{a\}$, $\{b\}$ or $\{a, b\}$. But the union or intersection of any combination of these is either $\emptyset$, $\{a\}$, $\{b\}$, or $\{a, b\}$. These are all in $\mathcal{T}_3$, as desired.* $\square$

3 Some set theory

The aim here is to introduce some notation and concepts from set theory that are ubiquitous in topology.

For the purposes of this class, a *set* will be any collection of objects. The objects in a set are called the *elements*. If X is a set and a is an element of X, then we write

$$a \in X.$$

If Y is another set with the property that $a \in Y \Rightarrow a \in X$, then we say that Y is a *subset of* X, and we write

$$Y \subset X, \quad \text{or} \quad Y \subseteq X.$$

(The symbols $\subset$ and $\subseteq$ will mean the same thing in this class.)

It is important to correctly distinguish between sets and elements, and the above notation helps with that. *Be sure to use the element $\in$ and subset $\subset$ notation correctly.* One reason this is so important is that, in topology, things that are subsets in one framework become elements in others. For example, if $\mathcal{T}$ is a topology on X, then the elements of $\mathcal{T}$ are subsets of X. Another example of this comes in the guise of the following definition.

Definition 2. *Suppose X is a set. The* **power set of** *X is the set $\mathcal{P}(X)$ consisting of all subsets of X. That is*

$$Y \in \mathcal{P}(X) \Leftrightarrow Y \subseteq X.$$

Written differently still,

$$\mathcal{P}(X) = \{Y \mid Y \subset X\}.$$

It turns out the power set plays a very interesting role in topology; this is explored in the next section.

3.1 A little more topology

Example 4. *If X is any set, then the power set $\mathcal{P}(X)$ is a topology on X. This is because $\emptyset, X \in \mathcal{P}(X)$, and the union and intersection of any subsets of X are again subsets of X (if this isn't obvious to you, you should convince yourself of it). The topology $\mathcal{P}(X)$ is called the* **discrete topology** *on X.*

The discrete topology is the topology with the most stuff in it (we will make this precise later). At the other extreme is the topology with the least stuff in it.

Example 5. *Let X be any set. Then the collection $\mathcal{T} = \{\emptyset, X\}$ is a topology on X by Remark 3. This topology is called the* **indiscrete topology** *or* **trivial topology** *on X.*

This shows that all sets have topologies, and they usually have at least two. (Fun exercise: Find all sets that have exactly one topology.)

3.2 Unions and intersections

Suppose U and V are sets. Their **union** is the set

$$U \cup V := \{x \mid x \in U \text{ or } x \in V\}.$$

> **Remark 4.** In this class, I will use the symbol := to mean that I am defining the thing appearing on the left of := in terms of the thing appearing on the right. So in its use above, the symbol $U \cup V$ should be thought of as the new thing, while it is assumed that the set of symbols $\{x \mid x \in U \text{ or } x \in V\}$ appearing on the right are understood by the reader.

Similarly, if we have a lot of sets, we can define their union as well. For example, if $U_1, U_2, \ldots, U_9$ are sets, then their union is

$$U_1 \cup U_2 \cup \ldots \cup U_9 := \{x \mid x \in U_1, \text{ or } x \in U_2, \text{ or } \ldots, \text{ or } x \in U_9\}.$$

A slightly less notationally cumbersome way of writing this exact same thing is as follows

$$\bigcup_{k=1}^{9} U_k = \{x \mid x \in U_1, \text{ or } x \in U_2, \text{ or } \ldots, \text{ or } x \in U_9\}.$$

Here is yet another way of writing it:

$$\bigcup_{k \in \{1,\ldots,9\}} U_k = \{x \mid \exists k \in \{1, \ldots, 9\}, x \in U_k\}.$$

Here we call $\{1, \ldots, 9\}$ the **indexing set**. Though you may not find this one less cumbersome, this latter notation has advantage that it emphasizes the indexing set, and in a way that does not depend on any special properties of the indexing set. Consequently, this notation generalizes to arbitrary indexing sets. Here is the most general notion of union that we will need in this class.

Definition 3 (Arbitrary Unions)**.** *Suppose A is a set, and that for each $\alpha \in A$ there is a set U_α. Then the* **union of the U_α (over $\alpha \in A$)** *is the set*

$$\bigcup_{\alpha \in A} U_\alpha := \{x \mid \exists \alpha \in A, x \in U_\alpha\}.$$

The set A is called the **indexing set***.*

Similarly, we can play the same game with intersection.

Definition 4 (Arbitrary Intersections)**.** *Suppose A is a set, and that for each $\alpha \in A$ there is a set U_α. Then the* **intersection of the U_α (over $\alpha \in A$)** *is the set*

$$\bigcap_{\alpha \in A} U_\alpha := \{x \mid \forall \alpha \in A, x \in U_\alpha\}.$$

The set A is called the **indexing set***.*

Note that the union $\cup$ is associated with the logical notions of 'or' and $\exists$, while the intersection $\cap$ is associated with 'and' and $\forall$. This suggests what language/proof strategies to use when proving statements involving $\cup$ and $\cap$.