# Differential Geometry - Homework 2

Due Monday, February 20

1. Consider $S^{2n-1}$ as a subset of $\mathbb{C}^n$, in the usual way. For each $z = (z_1, \ldots, z_n) \in S^{2n-1}$, define a curve $\gamma_z : \mathbb{R} \to S^{2n-1}$ by

$$\gamma_z(t) = (e^{it}z_1, \ldots, e^{it}z_n).$$

(a) Compute $\gamma_z$ in stereographic coordinates, and show that it is a smooth curve.

(b) Compute $\gamma_z'(t)$ in stereographic coordinates, and show that it is never zero.

(c) Show that $TS^1$ is diffeomorphic to $S^1 \times \mathbb{R}$.

2. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(r, \theta) = (r\cos(\theta), r\sin(\theta)).$$

Compute the matrix for $F_* : T_{(r,\theta)}\mathbb{R}^2 \to T_{F(r,\theta)}\mathbb{R}^2$ relative to the standard basis on the domain and codomain.

3. Consider the following vector fields on $\mathbb{R}^2$

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad W = (x^2 + y^2)\frac{\partial}{\partial x}$$

(a) For each of the above vector fields, compute its coordinate representation in polar coordinates on the right half-plane.

(b) Compute the Lie bracket $[V, W]$.

4. Suppose $V$ is a real vector space of dimension $n$, and consider the Grassmannian $G(r, V)$ of $r$-dimensional subspaces of $V$. Define

$$T := \sqcup_{W \in G(r,V)} W$$

to be the disjoint union of subspaces in the Grassmanian. Define a map $\pi : T \to G(r, V)$ by sending each $x \in W$ to $W$. Show that $T$ has a unique smooth manifold structure making it into a smooth rank $r$ vector bundle over $G(r, V)$, with $\pi$ the projection, and with the vector space structure on each fiber inherited from $V$.