Differential Geometry
Lecture 1

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1 Introduction

These are notes for Math 762, being taught at McMaster University during the Winter Term of 2017. Much of the initial content is taken from Lee's *Introduction to Smooth Manifolds*.

2 A quick review of topology

Recall that a *topological space* is a pair $(M, \mathcal{T})$ where M is a set, and $\mathcal{T} = \{U_i\}_{i \in I}$ is a collection of subsets $U_i \subseteq M$ (indexed by some indexing set I), satisfying the following axioms:

- $\emptyset, M \in \mathcal{T}$;
- $\cup_{i \in J} U_i \in \mathcal{T}$ for any subset $J \subset I$;
- $\cap_{i \in J} U_i \in \mathcal{T}$ for any finite subset $J \subset I$.

The $U_i \in \mathcal{T}$ are called the *open sets*. I will often write M in place of $(M, \mathcal{T})$, when the underlying topology $\mathcal{T}$ is understood.

A topological space M is called *Hausdorff* if for each $x, y \in M$ with $x \neq y$, there are open sets U, V with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. A topological space $(M, \mathcal{T})$ is called *second countable* if there is a countable subset $\mathcal{B} \subseteq \mathcal{T}$ of open sets so that each element of $\mathcal{T}$ can be written as a union of elements in $\mathcal{B}$.

2.1 Basic examples

Example 2.1. *Consider the case $M = \mathbb{R}^n$. For $x \in \mathbb{R}^n$, and $r \geq 0$, define the* open ball *by*

$$B_r(x) := \{y \in \mathbb{R}^n \mid \mathrm{dist}(x, y) < r\}.$$

Then define $\mathcal{T}_{st}$ to consist of the subsets of $\mathbb{R}^n$ that can be written as a union of open balls. One can check that $\mathcal{T}_{st}$ forms a topology that is Hausdorff and second countable (you should check this if this material is new). Then $\mathcal{T}_{st}$ is called the standard topology *on $\mathbb{R}^n$.*

Unless otherwise specified, we will always equip $\mathbb{R}^n$ with the standard topology. That said, the next example shows there are other topologies one can consider.

Example 2.2. *Consider $M = \mathbb{R}$. Define $\mathcal{T}_d$ to consist of all subsets of $\mathbb{R}$. This is a topology on $\mathbb{R}$ that is Hausdorff, but not second countable. We call T_d the* discrete topology*.*

Example 2.3. *Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x \neq 0$ and $x = y$. This induces, in the obvious way, an equivalence relation on the disjoint union $\mathbb{R} \sqcup \mathbb{R}$, and we let $M = (\mathbb{R} \sqcup \mathbb{R}) / \sim$ be the set of equivalence classes. Let $\pi : \mathbb{R} \sqcup \mathbb{R} \to M$ be the projection. Define $\mathcal{T}$ to be the set of subsets $U \subseteq M$ with the property that $\pi^{-1}(U)$ is open in $\mathbb{R} \sqcup \mathbb{R}$ (here $\mathbb{R}$ is equipped with the standard topology). Then $\mathcal{T}$ forms a topology that is second countable, but not Hausdorff.*

The remaining two examples give general ways of constructing new topological spaces from old ones.

Example 2.4. *Suppose $(M, \mathcal{T})$ is a topological space, and $S \subseteq M$ is a subset. Then we can define a topology $\mathcal{T}_S$ on S by taking $\mathcal{T}_S$ to consist of all unions of elements of the form $S \cap U$, where $U \in \mathcal{T}$. We will call $\mathcal{T}_S$ the* subspace topology*, and we will refer to $(S, \mathcal{T}_S)$ as a* subspace *of M.*

If M is Hausdorff (resp. second countable), then S is Hausdorff (resp. second countable).

Example 2.5. *Suppose M, M' are topological spaces. Define $\mathcal{T}_{M \times M'}$ to consist of all unions of subsets of $M \times M'$ of the form $U \times U'$, where U (resp. U') is open in M (resp. M'). Then $\mathcal{T}_{M \times M'}$ is a topology on $M \times M'$, called the* product topology*. If M, M' are Hausdorff (resp. second countable), then $M \times M'$ is as well.*

2.2 Continuous maps

Suppose M and M' are topological spaces. A function $f : M \to M'$ is called *continuous* if for each open $U' \subseteq M'$, the inverse image $f^{-1}(U')$ is open in M.

Example 2.6. *Suppose M is a topological space, and $S \subseteq M$ is a subspace. Let $\iota : S \to M$ be the inclusion map. Then ι is continuous.*

A function f is called a *homeomorphism* if it is bijective and f, f^{-1} are both continuous. More generally, we will say that $f : M \to M'$ is a *homeomorphism onto its image*, if the corestriction

$$f| : M \to f(M)$$

is a homeomorphism.

3 Smooth manifolds

Let M be a topological space. A *chart* on M is a pair (U, φ) with the property that $U \subseteq M$ is open, and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism onto its image. Given two charts (U_0, φ_0) and (U_1, φ_1), define the associated *transition function* by

$$\begin{aligned} \varphi_{10} : \varphi_0(U_0 \cap U_1) &\to \varphi_1(U_0 \cap U_1) \\ p &\mapsto \varphi_1(\varphi_0^{-1}(p)). \end{aligned}$$

Note that this is a function between open subsets of Euclidean space. We will say that (U_0, φ_0) and (U_1, φ_1) are *(smoothly-)compatible* if the transition function φ_{10} is a $\mathcal{C}^\infty$ map in the usual sense (i.e., all derivatives exist and are continuous).

A *(smooth) atlas* for M is a collection

$$\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$$

of charts that are compatible, and where the U_i cover M

$$M = \bigcup_{i \in I} U_i.$$

An atlas $\mathcal{A}$ is *maximal* if every chart compatible with all elements of $\mathcal{A}$ is already in $\mathcal{A}$. A maximal atlas is also called a *smooth structure*.

Definition 3.1. *A* (smooth) manifold *is a pair $(M, \mathcal{A})$, where M is a Hausdorff, second countable topological space, and $\mathcal{A}$ is a maximal (smooth) atlas.*

When the atlas is clear from context, I will generally write M instead of $(M, \mathcal{A})$.

Remark 3.2. *(a) More generally, we can define a $\mathcal{C}^\ell$-*manifold *similarly, by only requiring that the transition functions are $\mathcal{C}^\ell$, as opposed to $\mathcal{C}^\infty$.*

(b) The Hausdorff and second countable conditions are included to avoid certain pathological examples that would otherwise falsify theorems that will appear later.

I will often drop the word "smooth" in "smooth manifold", with the understanding that all manifolds are smooth, unless otherwise specified.

The next lemma shows that, when defining a smooth structure, it suffices to specify an atlas (as opposed to a *maximal* atlas).

Lemma 3.3. *Suppose $\mathcal{A}$ is an atlas for M. Then there is a unique maximal atlas $\overline{\mathcal{A}}$ with $\mathcal{A} \subseteq \overline{\mathcal{A}}$.*

We call $\overline{\mathcal{A}}$ the maximal atlas *induced* from $\mathcal{A}$.

Proof of Lemma 3.3. Given $\mathcal{A}$, define $\overline{\mathcal{A}}$ to consist of all charts on M that are compatible with all elements of $\mathcal{A}$. Clearly we have $\mathcal{A} \subseteq \overline{\mathcal{A}}$, from which it follows that the elements of $\overline{\mathcal{A}}$ cover M. We need to show the following:

- the charts in $\overline{\mathcal{A}}$ are compatible (so $\overline{\mathcal{A}}$ is an atlas);

- $\overline{\mathcal{A}}$ is maximal;
- if $\mathcal{B}$ is any other maximal atlas containing $\mathcal{A}$, then $\mathcal{B} = \overline{\mathcal{A}}$.

I will verify the first of these, and leave the other two to the reader.

To check compatibility, let (U_0, φ_0) and (U_1, φ_1) be charts in $\overline{\mathcal{A}}$, and let φ_{10} be the transition function. We need to show this is smooth. Since smoothness is a local property, it suffices to show that each point in $\varphi_0(U_0 \cap U_1)$ has a neighborhood on which φ_{10} is smooth. Towards this end, assume $p \in U_0 \cap U_1$. Then since $\mathcal{A}$ is an atlas, there is some chart $(U_2, \varphi_2) \in \mathcal{A}$ with $p \in U_2$. The definition of $\overline{\mathcal{A}}$ implies that the two transition functions

$$\varphi_{20} : \varphi_0(U_0 \cap U_2) \to \varphi_2(U_0 \cap U_2), \qquad \varphi_{12} : \varphi_2(U_2 \cap U_1) \to \varphi_1(U_2 \cap U_1)$$

are smooth. Hence the restricted composition

$$\varphi_{12} \circ \varphi_{20}| : \varphi_0(U_0 \cap U_1 \cap U_2) \to \varphi_1(U_0 \cap U_1 \cap U_2)$$

is smooth as well. Note that on $\varphi_0(U_0 \cap U_1 \cap U_2)$, we have

$$\varphi_{12} \circ \varphi_{20} = \varphi_1 \circ \varphi_2^{-1} \circ \varphi_2 \circ \varphi_0^{-1} = \varphi_1 \circ \varphi_0^{-1} = \varphi_{10}.$$

It follows that φ_{10} is smooth in a neighborhood of $\varphi_0(p)$, so we are done. $\square$

3.1 Basic examples

Example 3.1. *Consider $\mathbb{R}^n$ equipped with the standard topology. Define $\mathcal{A}$ to consist of the singleton $(\mathbb{R}^n, \mathrm{id})$, where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Then $\mathcal{A}$ is an atlas. However $\mathcal{A}$ is not maximal. Nevertheless, Lemma 3.3 says that $\mathcal{A}$ induces a unique maximal atlas $\overline{\mathcal{A}}$. We call $\overline{\mathcal{A}}$ the* standard *smooth structure on $\mathbb{R}^n$. It follows from Example 2.1 that $(\mathbb{R}^n, \overline{\mathcal{A}})$ is a smooth manifold.*

Example 3.2. *Consider $\mathbb{R}$ equipped with the discrete topology $\mathcal{T}_d$, from Example 2.2. I claim that there is a maximal atlas $\mathcal{A}$ on $(\mathbb{R}, \mathcal{T}_d)$ relative to which $(\mathbb{R}, \mathcal{A})$ satisfies all conditions of being a manifold, except the second countable condition. To see this, define $\mathcal{A}$ to consist of all pairs (U, φ), where U is any* point *in $\mathbb{R}$, and $\varphi : U \to \mathbb{R}^0$ is the obvious map (there is only one such map, since the domain and codomain are each single points). Note that U is in fact an allowable open set precisely because $\mathcal{T}_d$ is the discrete topology. It is easy to check that $\mathcal{A}$ is in fact an atlas (there are no non-trivial transition functions); in fact, it is a maximal atlas. The claim now follows.*

Example 3.3. *Let M be the topological space from Example 2.3. The standard smooth structure on $\mathbb{R}$ induces a maximal atlas $\mathcal{A}$ on M. However, $(M, \mathcal{A})$ is not a manifold because M is not Hausdorff.*

Example 3.4. *Suppose M is a manifold, and $U \subseteq M$ is an open subset. Equip U with the subspace topology, as in Example 2.4. Then we can define a smooth structure on U by intersecting all charts on M with U. We call U, equipped with this smooth structure, a* smooth open submanifold *of M.*

Example 3.5. *Suppose M, M' are smooth manifolds, and equip $M \times M'$ with the product topology, as in Example 2.5. Then we can define a smooth structure on $M \times M'$ by considering all products of charts for M and M'. We call this the* product smooth structure *on $M \times M'$.*