Differential Geometry
Lecture 13

David L. Duncan

We spent much of this lecture discussing category theory. The first part of our discussion was fairly standard and effectively covered the material of pp. 118-121 of Lee's book, and I will not rewrite that here. Our discussion culminated in the following.

Let $VECT_k$ denote the category of rank-k vector spaces and linear maps, and $VBund_k$ the category of rank-k vector bundles and bundle maps.

Proposition 0.1. *Suppose $\mathcal{F} : VECT_k \to VECT_\ell$ is a covariant functor with the property that*

$$\mathrm{GL}(\mathbb{R}^k) \to \mathrm{GL}(\mathcal{F}(\mathbb{R}^k)), \qquad f \mapsto \mathcal{F}(f)$$

is smooth. Then this induces a covariant functor $VBund_k \to VBund_\ell$ as follows: Suppose $\pi : E \to M$ is a rank-k vector bundle. Then

$$\mathcal{F}(E) := \sqcup_{p \in M} \mathcal{F}(E_p)$$

is naturally a smooth vector bundle on M. If $F : E \to E'$ is a vector bundle map, then

$$\mathcal{F}(F|_{E_p}) : \mathcal{F}(E_p) \to \mathcal{F}(E'_{F(p)}),$$

extends to a smooth bundle map $\mathcal{F}(F) : \mathcal{F}(E) \to \mathcal{F}(E')$.

A similar statement holds for covariant functors $\mathcal{F} : VECT_k \to VECT_\ell$.

Proof. We will prove that $\mathcal{F}(E)$ is a vector bundle; the remaining assertions are left for the reader. Let $\mu : \mathcal{F}(E) \to M$ be the projection. Fix local trivializations $\{(U_\alpha, \Phi_\alpha)\}$ for E, and assume these cover M. That is,

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

restricts to a linear map on each factor, and there is some $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$ so that

$$\Phi_\beta \circ \Phi_\alpha^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v).$$

Note that

$$\mu^{-1}(U_\alpha) = \sqcup_{p \in U_\alpha} \mathcal{F}(E_p).$$

Define $\Psi_\alpha : \mu^{-1}(U_\alpha) \to U_\alpha \times \mathcal{F}(\mathbb{R}^k)$ sending $v \in \mathcal{F}(E_p)$ to $\mathcal{F}(\Phi_\alpha(v)) \in \mathcal{F}(\{p\} \times \mathbb{R}^k)$. Then functoriality of $\mathcal{F}$ implies that on $U_\alpha \cap U_\beta$, these satisfy

$$\Psi_\beta \circ \Psi_\alpha^{-1}(p, v) = (p, \mathcal{F}(\tau_{\alpha\beta}(p))v),$$

where
$$\mathcal{F}(\tau_{\alpha\beta}(\cdot)) : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathcal{F}(\mathbb{R}^k)).$$
The assumption on the functor implies that this is a smooth map, and so the result follows by the Vector Bundle Construction Lemma in Lee's book. $\square$

Example 0.1. *Consider the dual functor*
$$\mathcal{F} : VECT_k \to VECT_k$$
given by sending V to its dual vector space $V^* := \mathrm{hom}(V, \mathbb{R})$, and f to its pullback. We claim that the map
$$\mathrm{GL}(\mathbb{R}^k) \to \mathrm{GL}((\mathbb{R}^k)^*), \qquad f \mapsto \mathcal{F}(f)$$
is smooth. To see this, note that relative to the standard basis we have a vector space isomorphism $(\mathbb{R}^k)^* \cong \mathbb{R}^k$. With this identification, the above map is just the transpose $f \mapsto f^T$, which is smooth. It follows from the above proposition that if $E \to M$ is a rank-k vector bundle, then
$$E^* := \mathcal{F}(E) \to M$$
is a rank-k vector bundle as well. We call this the dual vector bundle*.*

Suppose $E, E' \to M$ are vector bundles. By using similar arguments, it follows that
$$E \otimes E' := \sqcup_{p\in M} E_p \otimes E'_p, \qquad E \oplus E' := \sqcup_{p\in M} E_p \oplus E'_p$$
are naturally smooth vector bundles over M, called the *tensor* and *direct sum* of E and E'. By repeating this, we can form larger tensor products, such as $\otimes^k E \to M$ for $k \geq 1$.

We can also form the *alternating* and *symmetric product bundles*
$$\Lambda^k E := \sqcup_{p\in M} \Lambda^k E_p, \qquad \mathrm{Sym}^k E_p := \sqcup_{p\in M} \mathrm{Sym}^k E_p,$$
where $\Lambda^k V$ is the kth exterior product, and $\mathrm{Sym}^k V$ is the kth symmetric product of a vector space V. The vector space inclusions
$$\Lambda^k V \subset \otimes^k V, \qquad \mathrm{Sym}^k V \subset \otimes^k V$$
induce inclusions
$$\Lambda^k E \subset \otimes^k E, \qquad \mathrm{Sym}^k E \subset \otimes^k E.$$