Differential Geometry
Lecture 2

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1 Dimension

We say that a manifold is an *n-manifold*, or a *manifold of dimension* n if all charts take values in $\mathbb{R}^n$ (same n for all charts).

Example 1.1. *The manifold* $\mathbb{R}^n$ *is an* n*-manifold.*

Example 1.2. *The disjoint union* $M = \mathbb{R} \sqcup \mathbb{R}^2$ *is a smooth manifold, but it is not an* n*-manifold for any* n*.*

2 Some more basic examples

Example 2.1. *Consider the set* $M = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\right\}$ *equipped with the subspace topology. This has no smooth structure making it a manifold. Indeed, any such smooth structure would imply that there is a neighborhood of* $(0, 0)$ *in* M *that is homeomorphic to an interval, which is not true (any such neighborhood contains* $\times$*).*

Example 2.2. *Suppose* V *is a real vector space of (real) dimension* n*. Then* V *is naturally an* n*-manifold. Indeed, any choice of basis for* V *determines a vector space isomorphism* $\varphi : V \to \mathbb{R}^n$*. This gives an atlas for* V*, and hence a unique maximal atlas (i.e., smooth structure).*

Informally, the word 'naturally' that appears above means that the manifold structure is independent of choices (we will likely formalize this term later in the semester). It appears from the construction that the manifold structure just defined depends on the choice of isomorphism φ*. To finish the argument, we therefore need to check that the maximal atlas constructed is in fact independent of this choice. To see this, suppose* $\psi : V \to \mathbb{R}^n$ *is an isomorphism obtained by picking a different basis. Then* $\psi \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ *is a vector space isomorphism, and so is smooth. It follows that* $(\mathbb{R}^n, \varphi)$ *and* $(\mathbb{R}^n, \psi)$ *are compatible, and so belong to the same maximal atlas.*

Example 2.3. *Suppose* V *is a complex vector space of complex dimension* n*. Then* V *is also a real vector space, and its real dimension is* $2n$*. Hence* V *is naturally a* $2n$*-manifold. For example, there is an isomorphism* $\mathbb{C}^n \cong \mathbb{R}^{2n}$ *of real vector spaces.*

3 The n-sphere

Define the *n-sphere* by

$$S^n := \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1 \right\},$$

and equip this with the subspace topology. To define a smooth structure on this, let $N = (0, \ldots, 0, 1) \in S^n$ be the north pole, and set

$$U_N := S^n \setminus \{N\}.$$

Define

$$\begin{array}{rcl} \varphi_N : U_N & \longrightarrow & \mathbb{R}^n \\ (x_1, \ldots, x_{n+1}) & \longmapsto & \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n). \end{array}$$

Similarly, define $S = (0, \ldots, 0, -1)$, $U_S := S^n \setminus \{S\}$, and

$$\varphi_S : U_S \longrightarrow \mathbb{R}^n, \qquad \varphi_S(x) = \varphi_N(-x).$$

Each of φ_S, φ_N are homeomorphisms, and in the homework you are asked you to show that the transition function

$$\varphi_S \circ \varphi_N^{-1} : \varphi_N(U_N \cap U_S) \longrightarrow \varphi_S(U_N \cap U_S)$$

is smooth. The charts φ_S, φ_N are called *stereographic coordinates*.

Clearly U_N, U_S cover S^n, so the pair $(U_N, \varphi_N), (U_S, \varphi_S)$ determine a unique maximal atlas. We call this the *standard smooth structure* on S^n. Since S^n is a subspace of $\mathbb{R}^{n+1}$, it is automatically Hausdorff and second countable, so S^n, equipped with this smooth structure, is an n-manifold.

4 The Grassmannian

Let $k = \mathbb{R}$ or $\mathbb{C}$, and suppose V is a vector space over the field k with $\dim_k(V) = n$. Fix an integer $0 \leq r \leq n$. The *Grassmannian* is the set

$$G(r, V) := \{W \subseteq V \mid W \text{ is a subspace, } \dim_k(W) = r\}$$

of all r-dimensional subspaces of W. If $V = k^n$, then we will write

$$G(r, n) := G(r, k^n).$$

If $r = 1$, then we set

$$\mathbb{P}^{n-1} := G(1, n),$$

which is *projective $n-1$-space*.

Our goal is to construct an atlas on $G(r, V)$ that gives it the structure of a smooth manifold. It will follow from the construction (and Examples 2.2 and

2.3; see Remark 4.1) that it will have dimension $r(n-r)$ if $k = \mathbb{R}$, and $2r(n-r)$ if $k = \mathbb{C}$. This example is a bit different from the previous, since we have not yet specified a topology; part of our work, therefore, is to specify a topology and prove it is Hausdorff and second countable.

We begin by constructing charts; we will ultimately use these to define the topology. Toward this end, fix $P \in G(r, V)$, and let $Q \subseteq V$ be any complementary subspace; that is $V = P \oplus Q$. Note that $\dim_k(Q) = n - r$. Define

$$U_Q := \{W \in G(r, V) \mid W \cap Q = \{0\}\}.$$

Let $L(P, Q)$ be the space of k-linear maps $P \to Q$.

Remark 4.1. *Note that $L(P, Q)$ is a vector space, and any choice of basis provides an identification of $L(P, Q)$ with the space of $(n-r) \times r$-matrices; that is, with $k^{r(n-r)}$. However, it is not necessary to introduce a basis at this stage, so we will work with $L(P, Q)$.*

Define a function

$$\psi_{PQ} : L(P, Q) \longrightarrow U_Q$$

by sending a linear map A to its graph

$$\Gamma(A) := \{x + Ax \mid x \in P\}.$$

The map ψ_{PQ} is well-defined and bijective (you should check this); let

$$\varphi_{PQ} := \psi_{PQ}^{-1}$$

be its inverse. Ultimately, our atlas will be constructed from the charts (U_Q, φ_{PQ}). However, first we address the topology.

We define the topology on $G(r, V)$ to be the coarsest topology (fewest number of open sets) in which all of the φ_{PQ} are homeomorphisms. This means that $U \subset G(r, V)$ is open if and only if $\varphi_{PQ}(U \cap U_Q)$ is open for all P, Q as above.

Claim 1: This topology is Hausdorff.

To see this, fix $W, W' \in G(r, V)$ and assume $W \neq W'$. It is not hard to see that there is some subspace Q with $V = W \oplus Q$ and $V = W' \oplus Q$. In particular, $W, W' \in U_Q$. Since φ_{WQ} is a bijection, it follows that

$$\varphi_{WQ}(W) \neq \varphi_{WQ}(W').$$

We know that $L(W, Q)$ is Hausdorff, so there are disjoint open sets $U, U' \subset L(W, Q)$ containing $\varphi_{WQ}(W)$ and $\varphi_{WQ}(W')$, respectively. It follows that

$$\varphi_{WQ}^{-1}(U), \text{and} \qquad \varphi_{WQ}^{-1}(U')$$

are disjoint open sets containing W and W', respectively.

Claim 2: This topology is second countable.

It suffices to show that there are a finite set $\{Q_j\}_{j\in J}$ of $n-r$-dimensional subspaces, so that $G(r,V)$ is covered by

$$\left\{U_{Q_j}\right\}_{j\in J}$$

(why is this sufficient?). To show this, fix a basis $e_1,\ldots,e_n$ for V. Let J be the set of subsets $j\subset\{1,\ldots,n\}$ of size $n-r$. Then for each $j\in J$, define Q_j to be the span of the e_i for $i\in j$.

Example 4.1. *(a) Suppose $n=4$ and $r=2$. Then J contains six elements*

$$\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}$$

For example, if $j=\{1,2\}$, then Q_j is the span of e_1,e_2.

(b) Suppose $n=2$ and $r=1$. Then J contains two elements

$$\{1\}, \qquad \{2\}.$$

To finish the proof of Claim 2, we need to show that each $W\in G(r,V)$ is contained in one of these Q_j. To see this, fix a basis $w_1,\ldots,w_r$ for W. We may assume $0<r<n$, otherwise there is nothing to prove. Then the set

$$(w_1,\ldots,w_r;e_1,\ldots,e_n)$$

is linearly dependent in V. This implies there is some i_1 so that e_{i_1} is expressible as a linear combination of

$$(w_1,\ldots,w_r;e_1,\ldots,e_{i_1-1}).$$

If this is linearly independent, then we stop; otherwise, there is some $i_2>i_1$ so e_{i_2} is expressible as a linear combination of

$$(w_1,\ldots,w_r;e_1,\ldots,e_{i_1-1},e_{i_1+1},\ldots,e_{i_2-1}).$$

Repeating inductively, this will process terminate exactly when there are $n-r$ of the e_i vectors remaining. These remaining e_i vectors span some Q_j, and it follows from the construction that none of these remaining e_i vectors lies in W. Hence $W\in U_{Q_j}$. This finishes the proof of Claim 2.

Next time, we will investigate the transition functions associated with the charts (U_Q,φ_{PQ}).