Differential Geometry
Lecture 4

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1 Diffeomorphisms

We say that smooth manifolds M, M' are *diffeomorphic* if there is a diffeomorphism $f : M \to M'$. If this is the case, we will write $M \cong M'$.

Proposition 1.1. *The relation diffeomorphism relation $\cong$ is an equivalence relation.*

Proof. We need to show it the relation is reflexive, symmetric, and transitive. For this, use the following facts:

- The identity map $\mathrm{Id} : M \to M$ is a diffeomorphism.
- If f is a diffeomorphism, then f^{-1} is a diffeomorphism.
- The composition of two diffeomorphisms is a diffeomorphism when it is defined.

□

The next example shows that $\cong$ is a distinctly coarser relation than equality, thereby making it a more manageable relation to work with in practice.

Example 1.1. *(A funny smooth structure on $\mathbb{R}$) Consider $\mathbb{R}$ with the standard topology. Define a chart (U, φ) by taking $U = \mathbb{R}$ and $\varphi : U \to \mathbb{R}$ to be the map $\varphi(x) = x^3$. Then φ is a homeomorphism, so $\{(U, \varphi)\}$ is an atlas; let $\mathcal{A}_{x^3}$ be the associated maximal atlas.*

Let $\mathcal{A}_{st}$ be the atlas associated with the standard smooth structure. Then $\mathcal{A}_{st} \neq \mathcal{A}_{x^3}$ are not the same. Indeed, the charts $(U, \varphi) \in \mathcal{A}_{x^3}$ are $(U, \mathrm{Id}) \in \mathcal{A}_{st}$ are not compatible, since the transition function $\mathrm{Id} \circ \varphi^{-1}$ is not smooth (it is the map $x \mapsto x^{1/3}$, which is not differentiable at $x = 0$).

However, $(\mathbb{R}, \mathcal{A}_{x^3})$ is diffeomorphic to $(\mathbb{R}, \mathcal{A}_{st})$. To see this, consider the map

$$f : (\mathbb{R}, \mathcal{A}_{x^3}) \longrightarrow (\mathbb{R}, \mathcal{A}_{st})$$

given by $f(x) = x^3$ (of course, this is just φ). In the coordinates above, this is $\mathrm{Id} \circ f \circ \varphi^{-1} = \mathrm{Id}$, which is the identity and hence smooth. The inverse exists and is also the identity in the above local coordinates. Hence f is a diffeomorphism.

2 Lie groups

Suppose G is a group with multiplication $m : G \times G \to G$, and identity $e \in G$. Let $\iota : G \to G$ by the inverse map $\iota(x) = x^{-1}$.

Definition 2.1. *The group G is a* Lie group *if it has a smooth structure in which m and ι are smooth maps.*

2.1 Examples (and non-examples)

Example 2.1. *Suppose V is a vector space. Then V together with vector addition is a Lie group.*

Example 2.2. *Suppose V is a vector space, and consider the space $L(V, V)$ of linear maps from V to itself. Then function composition*

$$m : L(V, V) \times L(V, V) \to L(V, V), \qquad (A, B) \mapsto A \circ B$$

is a smooth associative map with identity. However, $L(V, V)$ is not a Lie group with this multiplication m, because not every element of $L(V, V)$ has a multiplicative inverse.

Example 2.3. *Consider the set*

$$\mathrm{GL}(\mathbb{R}^n) := \{A \in L(\mathbb{R}^n, \mathbb{R}^n) \mid \det(A) \neq 0\}$$

of $n \times n$ matrices with non-zero determinant. This is an open submanifold of $L(\mathbb{R}^n, \mathbb{R}^n)$ (why?), and the multiplication on $L(\mathbb{R}^n, \mathbb{R}^n)$ restricts to a map

$$m : \mathrm{GL}(\mathbb{R}^n) \times \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$$

that is smooth. Every element of $\mathrm{GL}(\mathbb{R}^n)$ has an inverse, so $\mathrm{GL}(\mathbb{R}^n)$ equipped with m is a group. The function

$$\mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n), \qquad A \mapsto A^{-1}$$

sending a matrix to its inverse is smooth (why?), so $\mathrm{GL}(\mathbb{R}^n)$ is a Lie group. We call $\mathrm{GL}(\mathbb{R}^n)$ the general linear group.

Example 2.4. *Consider $S^1 \subset \mathbb{C}$, equipped with the multiplication structure coming from complex multiplication on $\mathbb{C}$. Then S^1 is a Lie group.*

Example 2.5. *Let $\mathbb{H} \cong \mathbb{R}^4$ be the set of quaternions. Consider $S^3 \subset \mathbb{H}$, equipped with the multiplication structure coming from quaternionic multiplication. Then S^3 is a Lie group.*

Example 2.6. *Let $\mathbb{O} \cong \mathbb{R}^8$ be the set of octonians. Consider $S^7 \subset \mathbb{O}$, equipped with the multiplication structure coming from octonionic multiplication. This multiplication is smooth, but not associative. Hence, S^7 with this multiplication, does not form a Lie group.*

Remark 2.2. *It turns out that the only spheres that admit Lie group structures are* S^0, S^1*, and* S^3*; this can be shown using homology/cohomology.*

Example 2.7. *Suppose* G_1 *and* G_2 *are Lie groups. Then the product* $G_1 \times G_2$ *is also a Lie group (equipped with the product smooth structure and the product group structure).*

Example 2.8. *Define the* n-torus

$$T^n := S^1 \times S^1 \times \ldots \times S^1,$$

where S^1 *appears* n *times. This is a Lie group of dimension* n*.*