Differential Geometry
Lecture 8

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1 Tangent space for general manifolds

As mentioned in the last lecture, our goal is to understand the tangent space T_pM to a general manifold M. The intuition is that the tangent space at p should only depend on what is going on near p; that is, the tangent space should behave well under restriction to open sets. This is the intuition we have from standard tangent vectors on $\mathbb{R}^n$, but it is not at all clear that derivations on a general manifold M behave in any nice way relative to restriction. However, it turns out that they do.

To make this precise, suppose $U \subseteq M$ is an open set, and let $\iota : U \to M$ be the inclusion map (i.e., $\iota(p) = p$ for all $p \in U$). The main result of this lecture is the following.

Theorem 1.1. *For each $p \in U$, the pushforward $\iota_* : T_pU \to T_pM$ is an isomorphism of vector spaces.*

Before getting into the proof, let's discuss the usefulness. Suppose we want to understand T_pM. (For example, a target problem we can try to solve is to show that T_pM is n-dimensional if M is an n-manifold.) Fix any chart (U, φ) with $p \in U$. Then the above theorem gives an isomorphism

$$T_pM \cong T_pU.$$

On the other hand, $\varphi : U \to \varphi(U)$ is a diffeomorphism, so it follows from last lecture that the pushforward

$$\varphi_* : T_pU \to T_{\varphi(p)}\varphi(U)$$

is a linear isomorphism. By Theorem 1.1 again, but this time applied to $\varphi(U) \subseteq \mathbb{R}^n$, we have another linear isomorphism

$$T_{\varphi(p)}\varphi(U) \cong T_{\varphi(p)}\mathbb{R}^n.$$

Finally, from Lecture 6, we have a natural isomorphism

$$T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n.$$

In summary, we have the following.

Corollary 1.2. *Any choice of chart (U, φ) at $p \in M$ naturally induces a linear isomorphism*

$$T_pM \cong \mathbb{R}^n.$$

In some sense, this gives us a complete understanding of the tangent space T_pM, and completes our inquiry into why 'the space of derivations at p' deserves the name 'tangent space at p'. (This, for example, answers the target problem: If M is an n-manifold, then T_pM has dimension n for all $p \in M$.)

All that remains now is to prove Theorem 1.1.

Proof of Theorem 1.1. First we will show that ι_* is surjective. Let $V \in T_pM$. We want to show that there exists some $W \in T_pU$ with the property that

$$\iota_* W = V. \tag{1}$$

We begin by sketching the main strategy. Then we will show what is wrong with the strategy as stated, and finally give the corrected version.

Our aim is to use $V : \mathcal{C}^\infty(M) \to \mathbb{R}$ to define a linear function of the form $W : \mathcal{C}^\infty(U) \to \mathbb{R}$, and then show that W satisfies the Leibniz rule, and (1). Our strategy is to define W in such a way that (1) necessarily holds. Namely, given $f \in \mathcal{C}^\infty(U)$, assume there exists some $\widetilde{f} \in \mathcal{C}^\infty(M)$ that extends f in the sense that

$$\widetilde{f}|_U = f.$$

Then define

$$W(f) := V(\widetilde{f}).$$

Then the Leibniz rule for W follows from that for V. It seems to also satisfy (1): If $\widetilde{f} \in \mathcal{C}^\infty(M)$, then setting $f := \widetilde{f}|_U = \widetilde{f} \circ \iota$ gives

$$V(\widetilde{f}) = W(f) = W(\widetilde{f} \circ \iota) = (\iota_* W)(\widetilde{f}). \tag{2}$$

(Note the key property that is being used here is that pullback by ι is the same as restriction to U.)

So what is wrong with the above strategy? Two things:

(a) Given f, there may not exist an extension $\widetilde{f}$. For example, consider the case $M = \mathbb{R}$, and $U = \mathbb{R} \setminus \{0\}$. The function $f : U \to \mathbb{R}$ defined by $f(x) = x/|x|$ is smooth on U, but has no smooth (or even continuous!) extension to all of M.

(b) Even if an extension $\widetilde{f}$ does exist, there may be multiple such extensions, and it is not clear that the definition of W is independent of the choice of extension. (E.g., this independence of extension is crucial in the verification of line (2).)

It turns out that (b) is fine, and we will address it shortly. Issue (a), on the other hand, is a serious issue with the basic strategy just outlined. Our resolution will be to show that there is always a sufficiently small neighborhood

$B \subseteq U$ of p with the property that every smooth function on B has a smooth extension to all of M. This will alter our proof of (1) a bit, since it is effectively like replacing U by a smaller neighborhood where all smooth functions extend, and we need to justify why this replacement scheme is valid.

Now we will address this extension issue honestly. The most technical part of the proof comes in the guise of the following lemma, the proof of which we defer until the end.

Lemma 1.3. *Suppose $U \subseteq M$ is open, and $p \in M$. Then there is an open neighborhood $B \subseteq U$ of p, and a smooth function $\beta \in \mathcal{C}^\infty(M)$ satisfying*

$$\beta|_B = 1, \qquad \beta|_{M \backslash U} = 0.$$

We will refer to β as a *bump function*. With the use of a bump function, we can now address the extension question: Suppose $f \in \mathcal{C}^\infty(U)$ is any function. Let β be as in the lemma, and define $\widetilde{f}(x)$ to be $\beta(x)f(x)$ if $x \in U$, and $\widetilde{f}(x) = 0$ if $x \notin U$. Then $\widetilde{f}$ is a smooth function, and satisfies

$$\widetilde{f}|_B = f|_B. \tag{3}$$

We will refer to $\widetilde{f}$ as an *extension of f on B*.

Returning now to the proof of surjectivity, suppose we are given $V \in T_pM$. Then define $W : \mathcal{C}^\infty(U) \to \mathbb{R}$ by sending $f \in \mathcal{C}^\infty(U)$ to

$$W(f) := V(\widetilde{f}),$$

where $\widetilde{f} \in \mathcal{C}^\infty(M)$ is any extension of f on B (as we just saw, such things always exist).

To see that the definition of $W(f)$ is independent of the choice of extension $\widetilde{f}$ to B, suppose $\widetilde{g}$ is a second extension satisfying (3). We want to show $V(\widetilde{f}) = V(\widetilde{g})$. Set

$$h := \widetilde{f} - \widetilde{g}.$$

Note that $h|_B = 0$ is identically zero. Let $\beta' \in \mathcal{C}^\infty(M)$ be a bump function that is identically equal to 0 on $M \backslash B$, and identically equal to 1 on some open subset $B' \subset B$ (such a bump function exists by Lemma 1.3). Then these conditions imply that

$$h(x) = h(x)(1 - \beta(x))$$

for all $x \in M$. That is, $h = (1 - \beta)h$ as functions, and so

$$V(\widetilde{f} - \widetilde{g}) = V(h) = V((1 - \beta)h).$$

Notice that h and $1 - \beta$ both vanish at p. It then follows from Lemma 1.2 (ii) in Lecture 6 that

$$V((1 - \beta)h) = 0.$$

Hence $V(\widetilde{f} - \widetilde{g}) = 0$, so linearity gives $V(\widetilde{f}) = V(\widetilde{g})$. This shows that W is well-defined.

Now we need to check that W is a derivation, and satisfies (1). I will verify the latter of these, and leave the derivation derivation up to the reader. Let $q \in \mathcal{C}^\infty(M)$ be any function, and set $f := q \circ \iota \in \mathcal{C}^\infty(U)$. Then we have $W(f) = V(\widetilde{f})$, where $\widetilde{f}$ is any extension of f on B. On the other hand, we know that q is already an extension of f on B, so since we know we can use any extension in the definition of W, we have

$$W(f) = V(q).$$

Since $W(f) = (\iota_* W)(q)$, the identity (1) follows, and so ι_* is surjective.

We now need to prove that ι_* is injective. For this, suppose

$$\iota_* W = 0$$

for some $W \in T_p U$. We want to show $W = 0$, so it suffices to show that $W(f) = 0$ for all $f \in \mathcal{C}^\infty(U)$. To see this, fix f, and let $\widetilde{f}$ be any extension of f on B. Note that f and $\widetilde{f} \circ \iota$ are both extensions of f on B. Then $f - \widetilde{f} \circ \iota$ vanishes on B, so it follows from the above that

$$W(f) = W(\widetilde{f} \circ \iota).$$

On the other hand, we have $W(\widetilde{f} \circ \iota) = (\iota_* W)(\widetilde{f}) = 0$. Hence $W(f) = 0$, as desired. □

Proof of Lemma 1.3. The construction of β is carried out in several stages. First, consider the function $\gamma : \mathbb{R} \to \mathbb{R}$ defined by

$$\gamma(t) = \begin{cases} e^{-1/t^2} & t > 0 \\ 0 & t \le 0 \end{cases}.$$

The limit $\lim_{t \to 0^+} e^{-1/t^2} = 0$ is zero, so γ is continuous. Moreover, one can check that its nth derivative satisfies

$$\gamma^{(n)}(t) = \begin{cases} \frac{p_n(t)}{t^{2n}} e^{-1/t^2} & t > 0 \\ 0 & t \le 0 \end{cases}$$

for some polynomial p_n. It follows from l'Hospital's rule that this is continuous as well, and so $\gamma \in \mathcal{C}^\infty(\mathbb{R})$ is smooth.

Next, for $r > 0$, define $\alpha(t) := \gamma(t)\gamma(r^2 - t)$. This is smooth, non-negative, and vanishes identically outside of the interval $(0, r^2)$. Now, define

$$\delta(t) := \frac{1}{\int_{-\infty}^{\infty} \alpha(\tau) d\tau} \int_{-\infty}^{t} \alpha(\tau) d\tau.$$

This satisfies $0 \le \delta(t) \le 1$ for all t, and

$$\delta(t) = 0 \text{ for } t \le 0, \qquad \delta(t) = 1 \text{ for } t > r^2.$$

To extend this to $\mathbb{R}^n$, for $R > 0$, define $\Delta : \mathbb{R}^n \to \mathbb{R}$ by

$$\Delta(x) := 1 - \delta(|x|^2 - R^2).$$

This is smooth, between 0 and 1, and satisfies

$$\Delta|_{B_R(0)} = 1, \qquad \Delta|_{\mathbb{R}^n \setminus B_{\sqrt{R^2+r^2}}(0)} = 0,$$

where $B_s(x)$ denotes the ball of radius s centered at $x \in \mathbb{R}^n$.

Finally, given a chart $\varphi : U \to \mathbb{R}^n$ and $p \in U$, choose r, R small enough so that $B_{\sqrt{R^2+r^2}}(p) \subset \varphi(U)$. For $q \in U$, define

$$\beta(q) := \Delta(\varphi(q) - \varphi(p)),$$

and for $q \notin U$, define $\beta(q) := 0$. Then β is smooth and satisfies the conditions of Lemma 1.3. $\square$