On Isoperimetric Problems

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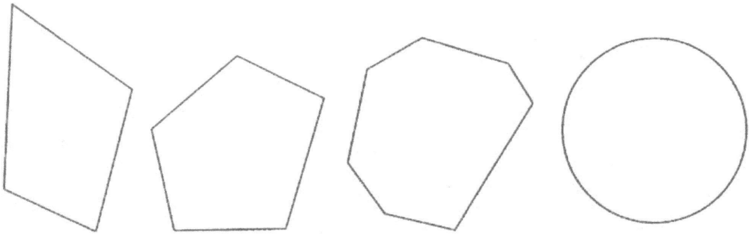
A Colloquium talk for Undergraduates

The Classical Isoperimetric Problem

Among all simple closed curves in the plane $\mathbb{R}^2$ with a given **perimeter** L, which has the largest **area** A?



- In $\mathbb{R}^3$, we ask for the solid body $\Omega \subset \mathbb{R}^3$ with given *surface area* which encloses the largest *volume*.
- One can ask the same question in $\mathbb{R}^d$, $d \geq 4$, by properly defining d-dimensional volume and $(d-1)$-dimensional surface area.
- This is a problem in shape optimization, part of the area of mathematics known as the Calculus of Variations.

Ancient Greece

For the ancients, mathematics meant geometry, and universal and/or optimal forms were revered.

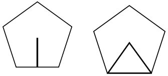
Theorem (Zenodorus (200–140 BCE))

- *Among* **regular polygons** *having the same perimeter, more sides yields more area enclosed.*
- *For any regular polygon, the circle with the same perimeter has greater area.*

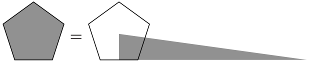
Zenodorus took it as obvious that a regular polygon contain more area than any other polygon with the same number of sides and the same perimeter. This is not difficult to show.

Zenodorus' method

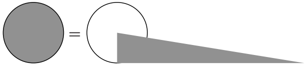
For a regular polygon, draw in the apothem, which is the perpendicular segment connecting the center to any side.
The apothem is the height of the isosceles triangle shown.



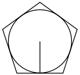
The area of the polygon is half of the apothem times perimeter:



The area of the circle is half of the radius times perimeter:



The inscribed circle has smaller perimeter but its radius = apothem of the polygon; the circle with the *same* perimeter will have a larger radius, hence larger area.



Jakob Steiner (1796–1863)

Steiner gave five different proofs of the following solution of the Isoperimetric problem:

Theorem (Steiner (1842))

If C is a simple closed curve in the plane and C is **not** *a circle, then there is a different simple closed curve C' with the same perimeter containing a larger area.*

- Steiner's methods are all geometrical and constructive: he assumes that C is a non-circular curve, then deforms it so that it increases the area.
- Each is based on the symmetries of the circle, mainly reflections in lines which divide the perimeter of the curve exactly in half.
- The device of comparing a domain with its reflection in a line or plane has been used in many contexts:
  - Alexandroff (1955): method of reflection in planes in differential geometry;
  - Serrin (1971), Gidas-Ni-Nirenberg (1979): extension of Alexandroff method to show symmetry of solutions to PDE
- Steiner neglected one detail: why does there exist a "best" isoperimetric curve at all?

The Isoperimetric Inequality

More modern treatments of the Isoperimetric Problem reformulate it as an inequality:

Theorem (Isoperimetric Inequality)

Let $\Omega \subset \mathbb{R}^2$ be a simple planar domain with perimeter L, containing area A. Then,

$$L^2 \geq 4\pi A,$$

with equality holding if and only if the boundary curve is a circle.

- The dual problem: the circle is also the curve which has the smallest perimeter containing a given area.
- The Isoperimetric Problem is rephrased as a problem in Calculus and Analysis!
- Parametrize the curve $C = \partial\Omega$ via $\vec{r}(t) = (x(t), y(t))$, $t \in [0, 1]$, with $\vec{r}(0) = \vec{r}(1)$.

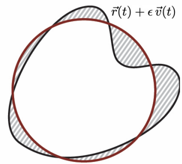
Then, by calculus and Green's Theorem,

$$L = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt, \qquad A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^1 [xy' - x'y]\,dt.$$

Hurwitz (1902) proved Isoperimetric Inequality using Fourier Series and Parseval's Identity.

The Calculus of Variations

Among all closed curves $\vec{r}(t) = (x(t), y(t))$, $t \in [0, 1]$, with given perimeter $L(\vec{r}) = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$, find the curve which maximizes area,
$A(\vec{r}) = \int_0^1 (xy' - x'y)\, dt$



Make a variation of the curve, $\vec{r}(t) + \epsilon\vec{v}(t)$, for $\epsilon > 0$. If $\vec{r}$ maximizes area among all curves, then it should be a maximum among these variations in ϵ.

Using the method of Lagrange Multipliers, there is a constant λ (the Lagrange multiplier!) for which a constrained maximizer of area $A(\vec{r})$ must satisfy:

$$\frac{d}{d\epsilon}\left[A(\vec{r} + \epsilon\vec{v}) - \lambda L(\vec{r} + \epsilon\vec{v})\right] = 0 \quad \text{when } \epsilon = 0.,$$

for any choice of variation curve $\vec{v}(t)$...

One may then derive the "critical point" equations for this maximization problem. They are a system of ODE, the Euler-Lagrange equations,

$$\frac{d}{dt}\left(y(t)+\lambda\frac{x'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}\right)=0,$$
$$\frac{d}{dt}\left(x(t)-\lambda\frac{y'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}\right)=0.$$

As $\vec{r}(t)$ may be parametrized by arclength (ie, speed $\sqrt{(x'(t))^2+(y'(t))^2}=1$), these equations may be simplified to

$$y+\lambda x'=b, \qquad x-\lambda y'=a, \qquad \text{for constants } a, b.$$

With $X=(x-a)$, $Y=(y-b)$, we get

$$Y=-\lambda X', \quad X=\lambda Y' \qquad \Longrightarrow \qquad X=\lambda\cos\frac{t-t_0}{\lambda}, \qquad Y=\lambda\sin\frac{t-t_0}{\lambda},$$

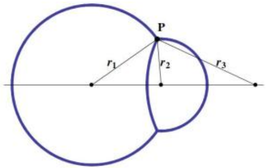
a circle, radius λ.

Karl Weierstrass (1879) proved the existence of maximizers using techniques of real analysis and the calculus of variations.

The Double Bubble

Here's a variation on the Isoperimetric Problem: find two disjoint regions Ω_1, Ω_2 in the plane, each of prescribed area, $A(\Omega_1) = a_1 > 0$, $A(\Omega_2) = a_2 > 0$, so that they are enclosed by curves of least total perimeter.

By placing them in contact, they may reduce combined perimeter



This problem was solved in 1993 by a team of undergraduate students at Williams College (Mass.)

Theorem (Alfaro, Brock, Foisy, Hodges, & Zimba)

For any given areas $a_1, a_2 > 0$, the two-region isoperimetric problem is solved by a unique "standard double-bubble": each boundary arc is **circular**, *and the arcs meet at an angle of 120º.*

This result was extended to any finite number of regions with prescribed areas, by Morgan & Wichiramala (2002).

The Triple Bubble!

This is a simulation of the "gradient flow" of the perimeter: in time it reduces perimeter to find the minimizer. (Chong Wang, 2018)

Inhibitory systems and patterns

Many models of pattern formation work through two competing mechanisms:

- "Activation", a term which is locally attractive which encourages concentration;
- "Inhibition", a term which is long range or non-local and which promotes fragmentation.

A Nonlocal Isoperimetric Problem

For any set $\Omega \subset \mathbb{R}^2$ with finite area and perimeter, define an energy,

$$E(\Omega) = \mathrm{Per}(\Omega) + \gamma \iint_\Omega \left[\iint_\Omega G(x,y)\, dx \right] dy, \qquad A(\Omega) = M$$

- The first term is perimeter, which we know is minimized for $\Omega =$ disk.
- The second term involves a Green's function, which is used in the solution of Laplace's equation.
- This term is non-local as it involves integration over the set Ω.
- As $G(x,y) \to +\infty$ as $y \to x$, this term is repulsive, it wants to split Ω into small pieces and move them far away from each other.
- The second term is **maximized** when $\Omega =$ disk.
- Which term wins when we minimize $E(\Omega)$ over all possible sets $\Omega \subset \mathbb{R}^2$?

Dilute mixtures, or the leopard's spots

Replace $\mathbb{R}^2$ by periodic boundary conditions on a large box (ie, a 2D torus.)
When the area of Ω is small, and γ is large, Choksi & Peletier (2010) proved that this happens:

Two-color spots

There is a form of energy for the 2-region Nonlocal Isoperimetric Problem [Ren-Wei (2012), A-Bronsard-Lu-Wang (2019)]