Stanley Alama / Research Description

My research is in elliptic partial differential equations and systems and in the calculus of variations. I am especially interested in finding solutions and mathematical clarifications of physical phenomena through rigorous methods of analysis.

Recently I have worked on Ginzburg–Landau equations and systems which describe phase transitions in high-temperature superconductors, superfluids, and liquid crystals. A Ginzburg–Landau (G–L) model is defined by a nonlinear functional for an complex scalar or vector "order parameter" $\Psi \in \mathbb{C}^m$ and a vector field A (the magnetic potential in superconductors). The functional takes the general form

$$E(\Psi, A) = \int_\Omega \left\{ |(\nabla - iA)\Psi|^2 + \kappa^2 F_{\text{pot}}(\Psi) + F_{\text{mag}}(\Psi, A) \right\} dx, \tag{GL}$$

where Ω is a domain in $\mathbb{R}^n$ $(n = 2, 3)$ (the physical sample or its cross-section) and the potential F_{pot} and field energy F_{mag} depend on the physical context. In the classical G–L model of superconductivity $\Psi \in \mathbb{C}$ is scalar, and the potentials are given by $F_{\text{pot}} = \frac{1}{2}(|\Psi|^2 - 1)^2$ and $F_{mag} = (\nabla \times A - h_{ex})^2$ with h_{ex} an external applied field.

In recent years G–L systems have provided a wonderful setting for new results in nonlinear analysis: the Euler–Lagrange equations are an elliptic system which regularizes a classical (but analytically difficult) harmonic map problem. In an appropriate singular limit the solutions develop singularities (such as vortices in the $U(1)$ model of superconductivity), and the presence, number, location and local structure of these singularities is an area of intensive mathematical activity.

My goal is to further develop this exciting interplay between analysis and physics, and to introduce new techniques in PDE, the calculus of variations, and nonlinear functional analysis which are both inspired by and shed new light upon these phenomena.

Brief summary of recent published work (1999-2004):

I. Pinning effects in Ginzburg–Landau models

In a series of papers ([1], [5], [6]) we study two-dimensional Ginzburg–Landau models for inhomogeneous or multiply connected superconductors and Bose–Einstein Condensates (BEC). An inhomogeneous superconductor is described variationally by (GL) for a scalar complex $\Psi \in \mathbb{C}$, with an inhomogeneous potential term,

$$F_{\text{pot}}(\Psi) = \frac{1}{4}\left(|\Psi|^2 - a(x)\right)^2,$$

with $a(x) \sim (T_c - T)$, with T the temperature and $T_c = T_c(x)$ the critical temperature for the onset of superconductivity in the material, non-constant to model impurities. For BEC, the unknown magnetic potential A is replaced by an imposed external rotation $A \to \omega x^\perp$, with given constant angular speed ω playing the role of the applied field in superconductivity.

Most previous mathematical work on pinning has been restricted to the case where $a(x) \geq 0$ in the sample domain Ω (see [ASaSe], [AnSh], [AnBP], for instance.) We consider the case where $a(x) < 0$ on smooth, open subdomains in Ω. In this situation, the pinning sites (where $a(x) < 0$) act as giant vortices, acquiring large degree for large but bounded (independent of κ) applied fields. For bounded fields the order parameter $|\Psi| \to \sqrt{a^+(x)}$ in the region where $a(x) > 0$, and there are no interior vortices. This result is similar to the behavior described in [AnBP] for $a(x)$ which vanish at isolated points in Ω.

The most interesting regime is when the applied field is allowed to grow with κ, $h_{ex} = O(\ln \kappa)$. We prove that there is a critical value of h_{ex} for which vortices first begin to appear in the region where $a(x) > 0$. Rather than cluster near the pinning sites, vortices will first nucleate in the interior, at points which are specifically identified via the solution of a singular elliptic boundary-value problem. This elliptic problem defines a sort of harmonic conjugate for the limiting problem (a weighted harmonic map functional), and some non-trivial regularity problems must be resolved near the boundaries of the pinning sites, where the equation loses uniform ellipticity. Similar phenomena also occur if we take a multiply-connected domain Ω with non-vanishing $a(x)$ (taken to be constant, for example.) This is because the negativity of a also leads to a multiply-connected domain for the limiting harmonic map problem which determines the vorticity of the giant vortices in the pinning sites.

We have also been working on a three-dimensional analogue for the BEC functional, with our postdoc J.A. Montero, which we will discuss in the following paragraph.

II. Vortices in 3D With our postdoc J.A. Montero we have some results on vortices in three-dimensional superconductors and BEC.

In our paper [4] we consider a three-dimensional solid Ω subjected to a constant applied field $\vec{h}_{ex} = \lambda \ln \kappa \, \hat{e}_3$ along the vertical axis. The energy functional is then:

$$E_\kappa(\Psi, A) = \int_\Omega \left\{ \frac{1}{2} |(\nabla - iA)\Psi|^2 + \frac{\kappa^2}{4} \left(|\Psi|^2 - 1 \right)^2 \right\} dx + \frac{1}{2} \int_{\mathbf{R}^3} \left| \nabla \times A - \vec{h}_{ex} \right|^2 dx.$$

Formally, we expect that vortices will first appear in Ω at a critical value λ^*, and that the vortices should be curves γ minimizing the limit functional (in the sense of Γ-convergence of de Giorgi),

$$G_\lambda(\gamma) = \int_\gamma ds - \lambda \int_\gamma B_0 \cdot d\vec{s},$$

where B_0 is a vector field satisfying $\nabla \times (\nabla \times B_0) = \hat{e}_3$, with $B_0 \times \nu = 0$ on $\partial \Omega$, ν the exterior unit normal. This limiting energy is formally obtained by expanding the energy about the vortexless "Meissner state" solution, and B_0 comes from a Hodge-type decomposition theorem.

The analysis of the limit process is possible because of the work of Jerrard & Soner [JS], who recognized that the important quantity in passing to the limit is the *weak Jacobian* of Ψ, $J(\Psi) = \frac{1}{2} \nabla \times \Im\{\bar{\Psi} \, \nabla \Psi\}$, which is weakly compact in a space of 1-currents. Thus the vortices associated to minimizing (Ψ, A) are not *a priori* curves, but integer rectifiable 1-currents. Connecting global minimizers of E_κ to minimizers of G_λ remains an open question, as we lack control of the norm of the Jacobians for fields of order $\ln\kappa$ where global minimizers first exhibit vortices in a general domain.

Instead, we use the method of [JMS] to constuct *local* minimizers to E_κ which converge (in the sense of weak Jacobians) to minimizers of G_λ. This method enables us to have concrete results on the critical λ^* in the case where $\Omega = B_R(0)$ is a solid ball. For the ball, we have an explicit λ^* (given as a function of the radius R) such that: if $\lambda < \lambda^*$, the global minimizer of G_λ is the zero current, that is, no vortex in the ball; if $\lambda > \lambda^*$, the global minimizer is the vertical diameter of the ball. We conjecture that $\lambda^* \ln\kappa$ gives the highest-order term in the expansion of the critical applied field, at which *global* minimizers first acquire vortices. Indeed, we also give a value λ^{**}, with $\lambda^{**} < \lambda^*$ but $\frac{\lambda^{**}}{\lambda^* \to 1}$ for radii $R \to \infty$, so that global minimizers have no vortices for $\lambda < \lambda^{**}$.

We are currently writing up a similar result for BEC where Ω is a circular torus in $\mathbf{R}^3$. This result combines some of the features of our work XXX on pinning and multiply connected superconductors with the 3-D superconductivity result mentioned above. The energy functional is now

$$E_\varepsilon(u) = \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 - \omega\,(-y, x, 0)\cdot \Im\{\bar u\,\nabla u\} + \frac{1}{4\varepsilon^2}(a(x) - |u|^2)^2 \right\} dx,$$

for complex-valued $u \in H_0^1(\Omega; \mathbb{C})$. Here ω is a (given) real angular speed, which plays the role of the applied field in the superconductivity model. As in our pinning problems, $a(x)$ is a smooth function which vanishes at $\partial\Omega$, which in this context represents the trapping potential by which the BEC is confined (via lasers.) The case of an ellipsoidal domain Ω was studied by [AfJ], [J2], who found that vortices are bent, and not straight. These bent vortices have been observed in experiments on BEC.

We follow the same scheme as in our paper [X??], and construct locally minimizing solutions to E_ε near the minimizers of the limiting line energy associated to this problem via Γ-convergence. Analysis of the limiting problem shows that vortices must bend, and are located in a region of a vertical cross-section strictly to the left of the center line. As in the 2-D problem, some interesting technical issues due to the vanishing of $a(x)$ near $\partial\Omega$ must be resolved. The tools developed in [J2] for the compactness of the Jacobians in the ellipsoidal case must be modified (and in some cases may be simplified.)

III. Spin-coupled systems. We consider a family of functionals (GL) for a complex vector order parameter, $\Psi : \Omega \to \mathbb{C}^2$ which carries both the usual information concerning superconductivity (the density of superconducting electrons and their currents) but also defines a *spin* $S = \Re\Psi \times \Im\Psi$. The potential and magnetic energies are $F_{\rm pot} = \frac{1}{2}(|\Psi|^2 - 1)^2 + \gamma|\psi_1^2 + \psi_2^2|^2$ and $F_{mag} = (\nabla \times A - h_{ex})^2 - 2\,g\,h_{ex}S$, with $g > 0$ the Zeeman coupling constant. These models have been proposed to study superconductors with ferromagnetic properties as well as certain Bose-Einstein condensates (BEC). The properties of minimizers of this functional depend strongly on the sign of the spin-coupling term: in "Phase I" (termed "ferromagnetic" in BEC) $\gamma > 0$ and the potential favors spins of constant size $\frac{1}{2}$, while in "Phase II" ("antiferromagnetic" in BEC) $0 < \gamma < 1$ and this term penalizes the spin vector.

•<u>*Half-integer vortices.*</u> The results for "Phase II" ($-1 < \gamma < 0$) are the most surprising. In this case, the singular limit of the energy leads to a harmonic map problem on a 2-torus, and the least energy solutions will carry two *half-integer* degrees. With L. Bronsard [3], [10] we studied a simplified Dirichlet problem in order to describe the local structure of vortices. We prove an asymptotic expansion of the energy in the spirit of Bethuel, Brezis, and Hélein [BBH] which shows that minimizers generally prefer the fractional degree to integer degree vortices. In subsequent work with Bronsard and P. Mironescu we have shown that the fractional degree vortices have nontrivial "spin polarized" cores: To resolve the singularity at the center of a half-degree vortex Ψ will rotate away from the limiting torus and acquire a non-zero spin rather than vanishing (as it does in the classical G-L vortices.) To do this, we blow up near a fractional degree vortex to obtain an entire, locally minimizing solution (in the sense of De Giorgi) to the G-L equations in all of $\mathbf{R}^2$, and use the local minimizing property to show that one component cannot vanish. It is still an open question whether the only solution to the limiting problem in $\mathbf{R}^2$ is the radially symmetric one.

•<u>*Critical fields with spin.*</u> In joint work with L. Bronsard ([8], [9]) we have studied the upper and lower critical fields in the "Phase I" regime in the singular limit as the Ginzburg–Landau parameter $\kappa \to \infty$. First, we show that the effect of spin coupling is to significantly decrease the lower critical field $H_{c1} = O(\ln \kappa)$, the smallest value of the applied field at which minimizers exhibit vortices. For large enough $g = O(\ln \kappa)$ we show the existence of a "spontaneous vortex state" whereby minimizers always have vortices for any applied field (and thus $H_{c1} = 0$.) In the second paper, we consider the transition to the normal state in high fields ($h_{ex} = O(\kappa^2)$.) We show that the critical field H_{c2}, the smallest applied field for which minimizers lose superconductivity in the interior of the sample, increases significantly with spin coupling. In fact, for sufficiently large $g = O(1)$ the upper critical field is also absent, and the normal state ($\Psi \equiv 0$) is never minimizing. These results are based on techniques of Sandier and Serfaty [SS1], [SS2], developed for the classical G–L model.

IV. The Lawrence–Doniach system. The Lawrence–Doniach system is a variational model for layered superconductors, and is commonly used by physicists to study many of the high-temperature superconductors. In this model the material is described by a coupled array of superconducting planes rather than as a three-dimensional solid. The functional (GL) is modified by defining Ψ as a sequence $(\psi_n(x))$ of complex order parameters, one for each superconducting plane, and by replacing z derivatives by gauge-invariant finite differences in the gradient term. Our recent results treat the situation where an external magnetic field is directed *parallel* to these planes.

•<u>*Vortex lattices.*</u> With L. Bronsard and A.J. Berlinsky [11], [13] we studied the structure of low-energy critical points of the Lawrence–Doniach functional in a "weak-coupling" limit. We consider two different settings: a bounded domain of rectangular cross-section, and planar solutions satisfying a doubly-periodic ansatz. In the periodic case [13] we must develop a new functional setting for the variational problem in order to eliminate the infinite-dimensional degeracy due to gauge invariance. In both cases, we use a Lyapunov–Schmidt decomposition to reduce to a finite-dimensional variational problem on a smooth Hilbert manifold. We provide a complete classification of all low energy solutions in the finite cross-section case. In the periodic setting energy minimization selects a *unique* optimal choice of period lattice from all possible geometries. Finally, we determine (in [11]) the range of validity of our perurbation approach (in terms of various parameters in the model) via *a priori* estimates on solutions and an original argument based on the implicit function theorem. One conclusion of this analysis is that the result seems appropriate for *large applied fields.*

•*Isolated interlayer vortices.* In work with L. Bronsard and E. Sandier [7] we studied the structure of *isolated* interlayer vortices in the Lawrence–Doniach model. Because of the half-discrete, half-continuous nature of the model, there can be no radially symmetric solutions to the Lawrence–Doniach equations, and therefore the problem of finding an isolated vortex solution in a parallel field is fully two-dimensional. Furthermore, the discreteness precludes an easy definition of "vortices" in terms of the degree of the order parameter. We use methods of Jerrard [J1] and Sandier [Sa] to locate the singularities and obtain uniform estimates away from them, in the continuum limit as the fundamental length scales tend to zero. These methods were derived for the classical G–L models, but have the advantage of being based entirely on the energy and do not rely on any higher regularity of solutions (which is not available for the mixed finite difference and differential equations setting of the LD model.)

Indeed, the local profile of the magnetic field induced by an interlayer vortex has been controversial in the physics literature, and our result refutes physisists' claim that Lawrence–Doniach vortices have a "nonlinear core".

With Sandier we are currently studying other limiting regimes with the goal of connecting the weak-coupling solutions of [11] with high field behavior in a small length scales limit as in [7]. The methods are as in [SS2], [SS3].

Earlier work (before 2000)

V. SYMMETRIC VORTICES. In joint work with L. Bronsard and T. Giorgi [15], [16], [17], we study isolated, radially symmetric vortices for the classical gauge-invariant Ginzburg–Landau model and for S. C. Zhang's $SO(5)$-model which unifies high-temperature superconductivity and antiferromagnetism.

•*Uniqueness.* For the classical G–L model, we give the first proof of *uniqueness* of symmetric vortex solutions [17], a long-standing open problem in the well-known reference text of Bethuel, Brezis, & Hélein [BBH]. In an interesting twist, we use the Mountain Pass Theorem to obtain this uniqueness result. Our result complements the recent uniqueness results of Pacard & Rivière [PR], who show that an entire degree-one solution of G–L must be radially symmetric.

•*Core structures in the $SO(5)$ model.* The $SO(5)$-model introduces a scalar SC order parameter ψ, a real antiferromagnetic order parameter $m \in \mathbb{R}^3$, and the usual magnetic potential A, with $F_{\text{pot}} = \frac{1}{2}(|\psi|^2+|m|^2-1)^2+g|m|^2$ and $F_{mag} = (\nabla\times A-h_{ex})^2$. (However, the gradient energy of m is not coupled to the magnetic field.) Since when $m = 0$ the $SO(5)$ energy reduces to the classical G–L energy we treat it as a perturbation problem and apply bifurcation theory to study the structure of symmetric vortices. We build on our uniqueness result to give compactness, bifurcation results, and an exact characterization of the solutions of the radially symmetric $SO(5)$ vortex system [15]. For these results we require energy-independent *a priori* estimates on solutions, topological arguments from bifurcation theory, and Struwe's min-max theory on convex sets in Banach spaces.

In continuing work with L. Bronsard and T. Giorgi we are looking at symmetric vortices in other gauge field models and at multi-vortex configurations in the $SO(5)$ model.

VI. Strongly indefinite variational problems: These projects concern semilinear elliptic equations with nonlinear potentials which change sign. These can be formulated variationally, but the associated functionals are strongly indefinite and classical methods based on convexity or monotonicity fail. On the other hand, these equations exhibit a rich variety of solutions and provide many very interesting new variational arguments and novel twists of the standard theory. For example, some of the techniques which we use in the uniqueness of vortex solutions described above appeared first in a simpler sub- and super-solution setting in the work described below.

•*Multiplicity for positive solutions.* Typically, we seek conditions on the indefinite nonlinearity which ensure that its negative part generates a bifurcation curve of local minimizers. Then we find a second solution created when the solution curve bends back under the influence of the positive part. In some situations the structure of the nonlinearity allows the branch to bend twice, giving a local minimizer separating a pair of (unstable) index-1 solutions. (See [18], [20], [21], [22], [25].) One motivation for studying such equations comes from geometrical PDE: when the nonlinearity has critical Sobolev growth (as in the prescribed scalar curvature equations, or the self-dual Chern–Simons models in dimension two,) local minimizers inherit stronger compactness properties and avoid blow-up problems which are common to these critical exponent problems (see [ES] and [25]).

•*C^1 vs. H^1 minimizers.* One important issue addressed in our work is the relationship between C^1-minimizers (obtained by the sub- and supersolution method or in bifurcation, for example) and H^1-minimizers (true minima among all admissible variations of energy) for elliptic equations with supercritical growth in the nonlinear term. The distinction between C^1 and H^1-minimizers was first remarked by Brezis & Nirenberg [BN], who proved that the two notions coincide for nonlinearities with subcritical or critical growth. In joint work with G. Tarantello [23], we present examples which demonstrate that, for nonlinearities with supercritical growth, C^1-minimizers may or may not be H^1-minimizers, depending on the explicit structure of the nonlinear terms.

VII. Heteroclinic solutions to PDE: We seek solutions to nonlinear elliptic problems on all of $\mathbf{R}^N$ with prescribed behavior at infinity. The technical difficulty in these problems is the loss of compactness due to translation invariance and loss of mass at infinity, which makes the direct application of global variational methods impossible. However, it is exactly because of this non-compactness that we find such a rich collection of solutions to our problem!

•*Multibump solutions.* In collaboration with Y. Y. Li, we consider nonlinear Schrödinger equations with a potential which is (asymptotically) periodic in space, and we construct "multibump" solutions, having nearly all of their mass concentrated in widely spaced packets [26]. The bumps are nearly identical copies of a "ground state" single-bump solution. The method of proof follows work by Séré [Sé] and Coti-Zelati & Rabinowitz [CZR] on heteroclinic solutions to Hamiltonian systems, and is based on P.L. Lions' concentration compactness principles and the construction of a topological min-max problem taylored to produce solutions with the desired shape.

•*Heteroclinic layers.* In a different setting, I have worked with L. Bronsard and C. Gui [19] on two-dimensional heteroclinic solutions to vector-valued PDE's, which arise in connection with some models of phase boundary motion. Because the conditions at infinity for this vector-valued PDE are themselves nontrivial heteroclinic solutions of ODE's, the desired solutions will have infinite energy over $\mathbf{R}^2$, and even after "renormalizing" the energy (in the spirit of work of Rabinowitz [Ra1] on heteroclinic solutions for reversible systems) we cannot use the direct method of minimization globally. Instead we solve problems in bounded domains and obtain uniform *a priori* estimates to pass to the limit as the bounded domains exhaust $\mathbf{R}^2$. More recently, other examples of heteroclinic-type solutions to PDE have been found by P. Rabinowitz [Ra2].

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Suggested letters of references: The following is a list of distinguished mathematicians who are well acquainted with my research, and could provide letters of reference.

Prof. Haïm Brezis, Université de Paris VI and Rutgers University.

Prof. Paul Rabinowitz, University of Wisconsin-Madison.

Prof. Robert V. Kohn, Courant Institute of Mathematical Science, New York University.

Prof. P. Bauman, Purdue University.

Prof. P. Sternberg, Indiana University.