1. Exercise 1.2.3 on Page 8:

List all possible arrangements of the four letters m, a, r, and y. Let C_1 be the collection of the arrangements in which y is in the last position. Let C_2 be the collection of the arrangements in which m is in the first position. Find the union and intersection of C_1 and C_2.
Answer:

$$C_1 = \{amry, army, mary, mray, ramy, rmay\} \qquad C_2 = \{mary, mayr, mray, mrya, myar, myra\}$$

$$C_1 \cup C_2 = \{amry, army, mary, mray, ramy, rmay, mray, mrya, myar, myra\}$$

$$C_1 \cap C_2 = \{mary, mray\}$$

2. Exercise 1.3.3 on Page 8:

A coin is to be tossed as many times as necessary to turn up one head. Thus the elements c of the sample space $\mathcal{C}$ are $H, TH, TTH, TTTH$, and so forth. Let the probability set function P assign to these elements the respective probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, and so forth. Show that $P(\mathcal{C}) = 1$. Let $C_1 = \{c \text{ is } H, TH, TTH, TTTH, \text{ or } TTTTH\}$. Compute $P(C_1)$. Next, $C_2 = \{c :$ c is $TTTTH$ or $TTTTTH\}$. Compute $P(C_2)$, $P(C_1 \cap C_2)$, and $P(C_1 \cup C_2)$.
Answer:
This is called a Negative Binomial distribution. Define a random variable X as the number of toss before a head.

$$P(X = k) = (\frac{1}{2})^k(1 - \frac{1}{2}) \qquad x \in \{0, 1, 2, ...\}$$

Therefore, the

$$\begin{aligned} P(\mathcal{C}) = P(X = \{0, 1, 2, ...\}) &= (1 - \frac{1}{2}) + (\frac{1}{2})^1(1 - \frac{1}{2}) + (\frac{1}{2})^2(1 - \frac{1}{2}) + ... \\ &= \frac{1}{2} \times \frac{1}{1 - \frac{1}{2}} \\ &= 1 \end{aligned}$$

Then,

$$P(C_1) = P(X = \{0, 1, 2, 3, 4\}) = \frac{1}{2} + \frac{1}{2}^2 + \frac{1}{2}^3 + \frac{1}{2}^4 + \frac{1}{2}^5 = \frac{1}{2} \times \frac{1 - \frac{1}{2}^5}{1 - \frac{1}{2}} = \frac{31}{32}$$

And

$$P(C_2) = P(X = \{4, 5\}) = \frac{1}{2}^5 + \frac{1}{2}^6 = \frac{3}{64}$$

$$P(C_1 \cap C_2) = P(X = \{4\}) = \frac{1}{2}^5 = \frac{1}{32} \qquad P(C_1 \cap C_2) = P(X = \{0, 1, 2, 3, 4, 5\}) = \frac{63}{64}$$

3. Exercise 1.3.10 on Page 19

A bowl contains 16 chips, of which 6 are red, 7 are white, and 3 are blue. If four chips are taken at random and without replacement, find the probability that:
(a) each of the four chips is red.
Answer(a):

$$P(\text{all red}) = \frac{\binom{6}{4}}{\binom{16}{4}} = \frac{3}{364} = 0.008241758$$

(b) none of four chips is red.
Answer(b):

$$P(\text{none red}) = \frac{\binom{10}{4}}{\binom{16}{4}} = \frac{21}{182} = 0.1153846$$

(c) there is at least one chip of each color.

$$P(\text{each color at least one}) = \frac{\binom{6}{2}\binom{7}{1}\binom{3}{1} + \binom{6}{1}\binom{7}{2}\binom{3}{1} + \binom{6}{1}\binom{7}{1}\binom{3}{2}}{\binom{16}{4}} = \frac{819}{1820} = 0.45$$

4. Exercise 1.4.3 on Page 28

Suppose we are playing draw poker. We are dealt (from a well-shuffled deck) five cards, which contain four spades and another card of a different suit. We decide to discard the card of a different suit and draw one card from the remaining cards to complete a flush in spades (All five cards spade). Determine the probability of completing the flush.
Answer:

$$Pr(\text{Flush}) = \frac{\binom{9}{1}}{\binom{47}{1}} = \frac{9}{47}$$

5. Exercise 1.4.8 on Page 28

In a certain factory, machines I, II and III are all producing springs of the same length. Machines I, II and II produce 1%, 4% and 2% defective springs, respectively. Of the total production of springs in the factory, Machine I produces 30%, Machine II produces 25%, and Machine III produces 45%.
(a) If one spring is selected at random from the total springs produced in a given day, determine the probability that it is defective.
Answer(a):

$$Pr(\text{defective}) = 0.3 \times 0.01 + 0.25 \times 0.04 + 0.45 \times 0.02 = 0.022$$

(b)Given that the selected spring is defective, find the conditional probability that it was produced by Machine II.

Answer(b):

$$Pr(\text{Machine II}|\text{Defective}) = \frac{0.25 \times 0.04}{0.022} = 0.4545$$

6. Exercise 1.5.2 on Page 39

For each of the following, find the constant c so that $p(x)$ satisfies the condition of being a p.m.f. of one random variable X.

(a) $p(x) = c(\frac{2}{3})^x$, $x = 1, 2, 3, ...$, zero elsewhere.

Answer(a):

$$\sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{\infty} c(\frac{2}{3})^x = c\frac{2}{3}\frac{1}{1-\frac{2}{3}} = 1 \qquad \rightarrow \qquad c = \frac{1}{2}$$

(b) $p(x) = cx, x = 1, 2, 3, 4, 5, 6$, zero elsewhere.

Answer(b):

$$\sum_{x=1}^{6} p(x) = \sum_{x=1}^{6} cx = 21c \qquad \rightarrow \qquad c = \frac{1}{21}$$

7. Exercise 1.6.3 on Page 43

Cast a die a number of independent times until a six appears on the up side of the side.

(a) Find the p.m.f. $p(x)$ of X, the number of casts needed to obtain that first six.

Answer(a):

This is a geometric distribution with $p = \frac{5}{6}$

$$p(x) = (\frac{5}{6})^{x-1}\frac{1}{6}$$

(b)Show that $\sum_{x=1}^{\infty} p(x) = 1$.

Answer(b):

$$\sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{\infty} (\frac{5}{6})^{x-1}\frac{1}{6} = \frac{1}{6}\frac{1}{1-\frac{5}{6}} = 1$$

(c)Determine $P(X = 1, 3, 5, 7, ...)$

Answer(c):

$$p(X = 1, 3, 5...) = \frac{1}{6}\frac{1}{1-\frac{25}{36}} = \frac{6}{11}$$

(d)Find the c.d.f. $F(x) = P(X \leq x)$.

Answer(d):

$$P(X \leq x) = 1 - (\frac{5}{6})^x$$

8. Exercise 1.7.5 on Page 50

Let the probability set function of the random variable X be

$$P_X(C) = \int_C e^{-x}dx, \quad \text{where} \quad \mathcal{C} = \{x : 0 < x < \infty\}$$

Let $C_k = \{x : 2 - \frac{1}{k} < x \leq 3\}, k = 1, 2, 3, ...$ Find the limits $\lim_{k\to\infty} C_k$ and $P_X(\lim_{k\to\infty} C_k)$. Find $P_X(C_k)$ and show that $\lim_{k\to\infty} P_X(C_k) = P_X(\lim_{k\to\infty} C_k)$.

Answer:

$$\lim_{k\to\infty} C_k = \lim_{k\to\infty}\{x : 2 - \frac{1}{k} < x \leq 3\} = \{x : 2 - \lim_{k\to\infty}\frac{1}{k} < x \leq 3\} = \{x : 2 < x \leq 3\}$$

$$P_X(\lim_{k\to\infty} C_k) = \int_2^3 e^{-x}dx = -e^{-x}|(x=3) - (-e^{-x}|(x=2)) = e^{-2} - e^{-3}$$

$$P_X(C_k) = \int_{C_k} e^{-x}dx = \int_{2-1/k}^{3} e^{-x}dx$$

$$= \int_2^3 e^{-x}dx = -e^{-x}|(x=3) - (-e^{-x}|(x=2-1/k)) = e^{-2+1/k} - e^{-3}$$

Therefore, we can get and prove that

$$\lim_{k\to\infty} P_X(C_k) = \lim_{k\to\infty}(\lim e^{-2+1/k} - e^{-3}) = e^{-2} - e^{-3} = P_X(\lim_{k\to\infty} C_k)$$

9. Exercise 1.8.2 on Page 56

Let X have the p.d.f. $f(x) = \frac{x+2}{18}, -2 < x < 4$, zero elsewhere. Find $\mathbb{E}(X)$, $\mathbb{E}[(X+2)^3]$, and $\mathbb{E}[6X - 2(X+2)^3]$.

Answer:

$$\mathbb{E}(X) = \int_{-2}^{4} xf(x)dx = \int_{-2}^{4} \frac{x^2+2x}{18}dx = \frac{1}{18}(\frac{1}{3}x^3 + x^2)|_{x=-2}^{x=4} = 2$$

$$\mathbb{E}[(X+2)^3] = \int_{-2}^{4}[(X+2)^3]f(x)dx = \int_0^6 \frac{1}{18}y^4dy = \frac{1}{90}y^5|_0^6 = 86.4$$

$$\mathbb{E}[6X - 2(X+2)^3] = 6\mathbb{E}(X) - 2\mathbb{E}[(X+2)^3] = -160.8$$

10. Exercise 1.8.11 on Page 57

Let X have the p.d.f. $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere.

(a) Compute $\mathbb{E}(X^3)$

Answer(a):

$$\mathbb{E}(X^3) = \int_0^1 3x^5 dx = \frac{1}{2}x^6|_0^1 = \frac{1}{2}$$

(b) Show that $Y = X^3$ has a uniform(0,1) distribution.

Answer(b):

$$J = \frac{dx}{dy} = \frac{dy^{1/3}}{dy} = \frac{1}{3}y^{-2/3}$$

$$\begin{aligned} f_Y(y) &= f_X(X = y^{1/3})|J| \\ &= 3(y^{1/3})^2 \times \frac{1}{3}y^{-2/3} \\ &= 1 \qquad y \in (0,1) \end{aligned}$$

Therefore, it follows a uniform (0,1) distribution.

(c) Compute $\mathbb{E}(Y)$ and compare this result with the answer obtained in part (a).

Answer(c):

$$\mathbb{E}(Y) = \int_0^1 y dy = \frac{1}{2}y^2|_0^1 = \frac{1}{2}$$

Therefore, the results are consistent.