1. Exercise 1.9.2 on Page 64

Let $p(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, ...$ zero elsewhere, be the p.m.f pf the random variable X. Find the m.g.f, the mean and the variance of X.

Answer:

$$\mathbb{E}(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}(\frac{1}{2})^x = \frac{1}{2e^{-t}-1} \tag{1}$$

$$\mu = m'(0) = -\frac{-2e^{-0}}{(2e^{-0}-1)^2} = 2 \tag{2}$$

$$\mathbb{E}(X^2) = m''(0) = \frac{2e^{-0}(2e^{-0}-1)^2 + 2e^{-t}(-2e^{-t})2(2e^{-t}-1)}{(2e^{-0}-1)^4}6 \tag{3}$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2 \tag{4}$$

2. Exercise 1.9.6 on Page 65

Let the random variable X have mean μ, standard deviation σ, and m.g.f $M(t)$, $-h < t < h$. Show that $\mathbb{E}(\frac{X-\mu}{\sigma}) = 0$, $\mathbb{E}((\frac{X-\mu}{\sigma})^2) = 1$, and $\mathbb{E}(\exp(t(\frac{X-\mu}{\sigma}))) = e^{-\mu t/\sigma}M(\frac{t}{\sigma})$, $-h\sigma < t < h\sigma$

Answer:

$$\mathbb{E}(\frac{X-\mu}{\sigma}) = \mathbb{E}(\frac{X}{\sigma}) - \mathbb{E}(\frac{\mu}{\sigma}) = \frac{\mathbb{E}(X)-\mu}{\sigma} = 0 \tag{5}$$

$$\mathbb{E}((\frac{X-\mu}{\sigma})^2) = \mathbb{E}(\frac{X^2 - 2X\mu + \mu^2}{\sigma^2}) = \frac{\mathbb{E}(X^2)-\mu^2}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1 \tag{6}$$

$$\mathbb{E}(\exp(t(\frac{X-\mu}{\sigma}))) = \mathbb{E}(e^{tX/\sigma}e^{-t\mu/\sigma}) = e^{-\mu t/\sigma}M(\frac{t}{\sigma}) \qquad -h\sigma < t < h\sigma \tag{7}$$

3. Exercise 1.9.14 on Page 66

Let X be a random variable with mean μ and variance σ^2 such that the third moment $\mathbb{E}((X-\mu)^3)$ about the vertical line through μ exists. The value of the ratio $\frac{\mathbb{E}(X-\mu)^3}{\sigma^3}$ is often used as a measure of skewness. Graph each of the following probability density functions and show that this measure is negative, zero, and positive for these respective distributions (which are said to be skewed to the left, not skewed, and skewed to the right, respectively).

(a) $f(x) = \frac{x+1}{2}$, $-1 < x < 1$, zero elsewhere.

Answer(a):

$$\mathbb{E}(X) = \int_{-1}^{1} \frac{x(x+1)}{2}dx = \frac{1}{2}(\frac{x^3}{3} + \frac{x^2}{2})|_{-1}^{1} = \frac{1}{3} \tag{8}$$

$$\mathbb{E}(X^2) = \int_{-1}^{1} \frac{x^2(x+1)}{2}dx = \frac{1}{2}(\frac{x^4}{4} + \frac{x^3}{3})|_{-1}^{1} = \frac{1}{3} \tag{9}$$

$$\mathbb{E}(X^3) = \int_{-1}^{1} \frac{x^3(x+1)}{2}dx = \frac{1}{2}(\frac{x^5}{5} + \frac{x^4}{4})|_{-1}^{1} = \frac{1}{5} \tag{10}$$

Then,

$$\mathbb{E}(\frac{(x-\mu)^3}{\sigma^2}) = \frac{\mathbb{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3} = \frac{-2\sqrt{2}}{5} \tag{11}$$

(b) $f(x) = \frac{1}{2}$, $-1 < x < 1$, zero elsewhere.

Answer(b):

$$\mathbb{E}(X) = \int_{-1}^{1} \frac{x}{2} dx = \frac{1}{2}(\frac{1}{2}x^2)|_{-1}^{1} = 0 \tag{12}$$

$$\mathbb{E}(X^2) = \int_{-1}^{1} \frac{x^2}{2} dx = \frac{1}{2}(\frac{1}{3}x^3)|_{-1}^{1} = \frac{1}{3} \tag{13}$$

$$\mathbb{E}(X^3) = \int_{-1}^{1} \frac{x^3}{2} dx = \frac{1}{2}(\frac{1}{4}x^4)|_{-1}^{1} = 0 \tag{14}$$

Then,

$$\mathbb{E}(\frac{(x-\mu)^3}{\sigma^2}) = \frac{\mathbb{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3} = 0 \tag{15}$$

(c) $f(x) = \frac{1-x}{2}$, $-1 < x < 1$, zero elsewhere.

Answer(c):

$$\mathbb{E}(X) = \int_{-1}^{1} \frac{x(-x+1)}{2} dx = \frac{1}{2}(-\frac{x^3}{3} + \frac{x^2}{2})|_{-1}^{1} = -\frac{1}{3} \tag{16}$$

$$\mathbb{E}(X^2) = \int_{-1}^{1} \frac{x^2(-x+1)}{2} dx = \frac{1}{2}(-\frac{x^4}{4} + \frac{x^3}{3})|_{-1}^{1} = \frac{1}{3} \tag{17}$$

$$\mathbb{E}(X^3) = \int_{-1}^{1} \frac{x^3(-x+1)}{2} dx = \frac{1}{2}(-\frac{x^5}{5} + \frac{x^4}{4})|_{-1}^{1} = -\frac{1}{5} \tag{18}$$

Then,

$$\mathbb{E}(\frac{(x-\mu)^3}{\sigma^2}) = \frac{\mathbb{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3} = \frac{2\sqrt{2}}{5} \tag{19}$$

4. Exercise 1.9.26 on Page 67

Let X have the exponential pdf, $f(x) = \beta^{-1}\exp(x/\beta)$, $0 < x < \infty$, zero elsewhere. Find the mgf, the mean, and the variance of X.

Answer:

$$\mathbb{E}(e^{tX}) = \int_{0}^{\infty} \beta^{-1}\exp(x/\beta)\exp(tx) = \frac{1}{1-\beta t} \tag{20}$$

Therefore,

$$\begin{aligned} M'(t) &= \frac{\beta}{(1-\beta t)^2}, \qquad M'(0) = \beta \\ M''(t) &= \frac{2\beta^2}{(1-\beta t)^3}, \qquad M''(0) = 2\beta^2 \\ \mathbb{E}(X) &= \beta^2 \end{aligned} \tag{21}$$

5. Exercise 2.1.6 on Page 83

Let $f(x,y) = e^{-x-y}, 0 < x < \infty, 0 < y < \infty$, zero elsewhere, be the pdf of X and Y. Then if $Z = X + Y$, compute $P(Z \leq 0)$, $P(Z \leq 6)$, and, more generally, $P(Z \leq z)$, for $0 < z < \infty$. What is the pdf of Z?

Answer:

$$F_z(Z) = \int_0^z \int_0^{z-x} e^{-x-y} dy dx = 1 - e^{-z} - ze^{-z} \tag{22}$$

Then, we have

$$P(Z \leq 0) = 0 \qquad P(Z \leq 6) = 1 - e^{-6} - 6e^{-6} \tag{23}$$

6. Exercise 2.1.13 on Page 84

Let X1,X2 be two random variables with the joint pdf $f(x_1, x_2) = 4x_1x_2$, for $0 < x_1 < 1$, $0 < x_2 < 1$, zero elsewhere. Compute $\mathbb{E}(X_1)$, $\mathbb{E}(X_1^2)$, $\mathbb{E}(X_2)\mathbb{E}(X_2^2)$, and $\mathbb{E}(X_1|X_2)$. Is $\mathbb{E}(X_1|X_2) = \mathbb{E}(X_1)\mathbb{E}(X_2)$? Find $\mathbb{E}(3X_2 - 2X_1^2 + 6X_1X_2)$.

Answer:

$$\begin{aligned}
\mathbb{E}(X_1) &= \int_0^1 \int_0^1 x_1 \times 4x_1x_2dx_2dx_1 = \frac{2}{3} \\
\mathbb{E}(X_1^2) &= \int_0^1 \int_0^1 x_1^2 \times 4x_1x_2dx_2dx_1 = \frac{1}{2} \\
\mathbb{E}(X_2) &= \int_0^1 \int_0^1 x_2 \times 4x_1x_2dx_2dx_1 = \frac{2}{3} \\
\mathbb{E}(X_2^2) &= \int_0^1 \int_0^1 x_2^2 \times 4x_1x_2dx_2dx_1 = \frac{1}{2} \\
\mathbb{E}(X_1X_2) &= \int_0^1 \int_0^1 x_1x_2 \times 4x_1x_2dx_2dx_1 = \frac{4}{9} = \mathbb{E}(X_1)\mathbb{E}(X_2) \\
\mathbb{E}(3X_2 - 2X_1^2 + 6X_1X_2) &= \frac{11}{3}
\end{aligned} \tag{24}$$

7. Exercise 2.2.3 on Page 92

Let X_1 and X_2 have the joint pdf $h(x_1, x_2) = 2e{-}x_1 - x_2, 0 < x_1 < x_2 < \infty$, zero elsewhere. Find the joint pdf of $Y_1 = 2X_1$ and $Y_2 = X_2 - X_1$.

Answer:

$$X_1 = \frac{Y_1}{2} \qquad X_2 = Y_2 + \frac{Y_1}{2} \tag{25}$$

$$J = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = \frac{1}{2} \tag{26}$$

$$f(y_1, y_2) = h(\frac{y_1}{2}, \frac{y_1 + 2y_2}{2})|J| = e^{-y_1-y_2} \tag{27}$$

8. Exercise 2.3.1 on Page 100

Let X1 and X2 have the joint pdf $f(x_1, x_2) = x_1 + x_2, 0 < x_1 < 1, 0 < x_2 < 1$, zero elsewhere. Find the conditional mean and variance of X2, given $X_1 = x_1, 0 < x_1 < 1$.
Answer:

$$\begin{aligned} f_{X_1}(x_1) &= \int_0^1 x_1 + x_2 dx_2 = x_1 + \frac{1}{2} \\ f_{X_2|X_2}(x_2|x_1) &= \frac{2(x_1 + x_2)}{2x_1 + 1} \end{aligned} \tag{28}$$

Therefore, the expectations:

$$\begin{aligned} \mathbb{E}(X_2|X_1 = x_1) &= \int_0^1 x_2 \frac{2(x_1 + x)2}{2x_1 + 1} dx_2 = \frac{3x_1 + 2}{6x_1 + 3} \\ \mathbb{E}(X_2^2|X_1 = x_1) &= \int_0^1 x_2^2 \frac{2(x_1 + x)2}{2x_1 + 1} dx_2 = \frac{4x_1 + 3}{12x_1 + 6} \\ \mathbb{V}(X_2|X_1 = x_1) &= \frac{6x_1^2 + 6x_1 + 1}{2(6x_1 + 3)^2} \end{aligned} \tag{29}$$

9. Exercise 2.3.7 on Page 101

Suppose X1 and X2 are discrete random variables which have the joint pmf $p(x_1 1, x_2) = (3x_1 + x_2)/24$, $(x_1, x_2) = (1, 1), (1, 2), (2, 1), (2, 2)$, zero elsewhere. Find the conditional mean $E(X_2|x_1)$, when $x_1 = 1$.
Answer:

$$\begin{aligned} p(x_1) &= \frac{4}{24} + \frac{5}{24} = \frac{9}{24} \\ \mathbb{E}(X_2|X_1 = x_1) &= \sum_{x_2} x_2 p(x_2|X_1 = x_1) = \frac{14}{9} \end{aligned} \tag{30}$$