1. Exercise 2.4.2 on Page 108

Let X and Y have the joint pmf described as follows:

(x,y)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
$p(x,y)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

and $p(x,y)$ is equal to zero elsewhere.

(a) Find the means μ_1 and μ_2, the variances σ_1^2 and σ_2^2, and the correlation coefficient ρ.

Answer(a):

$$\mu_1 = \frac{7}{5} \qquad \mu_2 = \frac{34}{15}$$

$$\sigma_1^2 = \frac{6}{25} \qquad \sigma_2^2 = \frac{134}{225} \qquad \rho = 0.2469$$

(b) Compute $\mathbb{E}(Y|X=1)$, $\mathbb{E}(Y|X=2)$, and the line $\mu_2 + \rho(\sigma_2/\sigma_1)(x-\mu_1)$. Do the points $[k, \mathbb{E}(Y|X=k)], k=1,2$, lie on this line?

Answer(b):

$$\mathbb{E}(Y|X=1) = (1 \times \frac{2}{15} + 2 \times \frac{4}{15} + 3 \times \frac{3}{15})/(\frac{9}{15}) = \frac{19}{9}$$

$$\mathbb{E}(Y|X=2) = (1 \times \frac{1}{15} + 2 \times \frac{1}{15} + 3 \times \frac{4}{15})/(\frac{6}{15}) = \frac{5}{2}$$

Next, the line $\mu_2 + \rho(\sigma_2/\sigma_1)(x-\mu_1)$ is

$$y = \frac{34}{15} + 0.2469 \times \sqrt{\frac{134/225}{6/25}} \times (x - \frac{7}{5}) = 0.389x + 1.722 = 2.111$$

The points are on the line.

$$1 \times 0.389 + 1.722 = \frac{19}{9}$$

$$2 \times 0.389 + 1.722 = 2.5$$

2. Exercise 2.4.3 on Page 108

Let $f(x,y) = 2$, $0 < x < y$, $0 < y < 1$, zero elsewhere, be the joint p.d.f. of X and Y. Show that the conditional means are, respectively. $(1+x)/2$, $0 < x < 1$, and $y/2$, $0 < y < 1$. Show that the correlation coefficient of X and Y is $\phi = \frac{1}{2}$.

Answer:

First, find the marginal:

$$f_X(x) = \int_x^1 2dy = 2(1-x)$$

$$f_Y(y) = \int_0^y 2dx = 2y$$

Then, the conditional distributions are given as:

$$f(x|y) = 2/(2y) = \frac{1}{y}$$

$$f(y|x) = \frac{1}{1-x}$$

$$\mu_{X|Y} = \int_0^y x\frac{1}{y}dx = \frac{y}{2}$$

$$\mu_{Y|X} = \int_x^1 y\frac{1}{1-x}dy = \frac{1+x}{2}$$

Now, we want to find the correlation coefficient. First,

$$\mu_1 = \int_0^1 x2(1-x)dx = \frac{1}{3}$$

$$\mu_2 = \int_0^1 y2ydy = \frac{2}{3}$$

$$\mathbb{E}(X^2) = \int_0^1 x^2 2(1-x)dx = \frac{1}{6}$$

$$\mathbb{E}(Y^2) = \int_0^1 y^2 2ydx = \frac{1}{2}$$

Therefore,

$$\sigma_1^2 = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{18}$$

$$\sigma_2^2 = \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 = \frac{1}{18}$$

Then,

$$\mathbb{E}(XY) = \int_0^1 \int_0^y 2xydxdy = \frac{1}{4}$$

$$\phi = \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sigma_x\sigma_y} = \frac{1}{2}$$

3. Exercise 2.4.6 on Page 109

Let X and Y have the joint p.d.f. $f(x,y) = 1$, $-x < y < x$, $0 < x < 1$, zero elsewhere. Show that, on the set of positive probability density, the graph of $\mathbb{E}(Y|x)$ is a straight line, whereas that of $\mathbb{E}(X|y)$ is not a straight line.

Answer:

The marginal distributions are:

$$f_X(x) = \int_{-x}^x 1dy = 2x$$

When $y > 0$, we have,

$$f_Y(y) = \int_y^1 1dx = (1-y)$$

When $y < 0$, we have,

$$f_Y(y) = \int_0^1 1dx = 1$$

Therefore,

$$f(x|y) = \frac{1}{1-\max(y,0)} \qquad -1 < -x < y < x < 1$$

$$f(y|x) = \frac{1}{2x} \qquad 0 < x < 1$$

The conditional means:

$$\mathbb{E}(X|Y) = \int_{\max(y,0)}^{1} x\frac{1}{1-\max(y,0)}dx = \frac{1+\max(y,0)}{2}$$

$$\mathbb{E}(Y|X) = \int_{-x}^{x} y\frac{1}{2x}dy = 0$$

Therefore, $\mathbb{E}(X|Y)$ is a straight line while $\mathbb{E}(Y|X)$ is not.

Note that the set of positive probability is indicating the space of probability is positive, which means we can not have negative values for pdf. In which case, the support should be determined correspondingly. However, this case we have$-1 < -x < y < x < 1$ and $0 < x < 1$. Many student write $y > 0$, which is not the case asked by the question.

4. Exercise 2.4.8 on Page 109

Let $\psi(t_1, t_2) = \log(M(t_1, t_2))$, where $M(t_1, t_2)$ is the mgf of X and Y. Show that

$$\frac{\partial \psi(0,0)}{\partial t_i} \qquad \frac{\partial^2 \psi(0,0)}{\partial t_i^2} \qquad i = 1, 2$$

and

$$\frac{\partial^2 \psi(0,0)}{\partial t_1 \partial t_2}$$

yield the means, the variances, and the co-variance of the two random variables. Use this result to find the means, the variances, the co-variance of X and Y of Example 2.4.4.

Answer:

Proof:

$$\frac{\partial \psi(0,0)}{\partial t_i} = \frac{1}{M(0,0)}\frac{\partial M(0,0)}{\partial t_i} = \frac{\partial M(0,0)}{\partial t_i} = \text{mean}$$

Calculations:

$$\frac{\partial^2 \psi(t_1,t_2)}{\partial t_i^2} = \frac{1}{M(t_1,t_2)}\frac{\partial^2 M(t_1,t_2)}{\partial t_i^2} - (\frac{1}{M(t_1,t_2)})^2(\frac{\partial M(t_1,t_2)}{\partial t_i})^2$$

$$\frac{\partial^2 \psi(0,0)}{\partial t_i^2} = \frac{1}{M(0,0)}\frac{\partial^2 M(0,0)}{\partial t_i^2} - (\frac{1}{M(0,0)})^2(\frac{\partial M(0,0)}{\partial t_i})^2 = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = \text{variance}$$

Then,

$$\frac{\partial^2\psi(t_1,t_2)}{\partial t_1\partial t_2} = \frac{1}{M(t_1,t_2)}\frac{\partial^2 M(t_1,t_2)}{\partial t_1\partial t_2} - (\frac{1}{M(t_1,t_2)})^2\frac{\partial M(t_1,t_2)}{\partial t_1}\frac{\partial M(t_1,t_2)}{\partial t_2}$$

Then,

$$\frac{\partial^2\psi(0,0)}{\partial t_1\partial t_2} = \mathbb{E}(XY) - \mu_x\mu_y$$

Lastly, Mgf given:

$$M(t_1,t_2) = \frac{1}{(1-t_1-t_2)(1-t_2)}$$

and

$$\psi(t_1,t_2) = -\log(1-t_1-t_2) - \log(1-t_2)$$

Next, we have

$$\frac{\partial\psi(t_1,t_2)}{\partial t_1} = \frac{1}{1-t_1-t_2}$$
$$\frac{\partial\psi(t_1,t_2)}{\partial t_2} = \frac{1}{1-t_1-t_2} + \frac{1}{1-t_2}$$
$$\frac{\partial^2\psi(t_1,t_2)}{\partial t_1^2} = \frac{1}{(1-t_1-t_2)^2}$$
$$\frac{\partial^2\psi(t_1,t_2)}{\partial t_2^2} = \frac{1}{(1-t_1-t_2)^2} + \frac{1}{(1-t_2)^2}$$
$$\frac{\partial^2\psi(t_1,t_2)}{\partial t_1\partial t_2} = \frac{1}{(1-t_1-t_2)^2}$$

Thus, using the formulas above, we have

$$\mu_1 = 1, \qquad \mu_2 = 2 \qquad \sigma_1^2 = 1 \qquad \sigma_2^2 = 2 \qquad Cov(X,Y) = 1$$

5. Exercise 2.5.1 on Page 116

Show that the random variables X_1 and X_2 with joint pdf

$$f(x_1,x_2) = \begin{cases} 12x_1x_2(1-x_2) & 0 < x_1 < 1, \qquad 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

are independent.

Answer:

$$f_{X_1}(x_1) = \int_0^1 12x_1x_2(1-x_2)dx_2 = 2x_1$$
$$f_{X_2}(x_2) = \int_0^1 12x_1x_2(1-x_2)dx_1 = 6x_2(1-x_2)$$

Since $f(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$, we can conclude that X_1 and X_2 are independent.

6. Exercise 2.5.4 on Page 116

Find $P(0 < X_1 < \frac{1}{3}, 0 < X_2 < \frac{1}{3})$ if the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = 4x_1(1 - x_2), 0 < x_1 < 1, 0 < x_2 < 1$,zero elsewhere.

Answer:

$$P(0 < X_1 < \frac{1}{3}, 0 < X_2 < \frac{1}{3}) = \int_0^{1/3} \int_0^{1/3} 4x_1(1 - x_2)dx_1dx_2$$
$$= \int_0^{1/3} \frac{2}{9}(1 - x_2)dx_2 \qquad = \frac{5}{81}$$

7. Exercise 2.5.8 on Page 116

Let X and Y have the joint pdf $f(x, y) = 3x$, $0 < y < x < 1$, zero elsewhere. Are X and Y independent? If not, find $\mathbb{E}(X|y)$.

Answer:

First, they can not be independent as $0 < y < x < 1$.

Then the marginal distribution of y:

$$f(y) = \int_y^1 3xdx = \frac{3}{2}(1 - y^2)$$

Then,

$$f(X|y) = \frac{2x}{1 - y^2}$$

Therefore, we have

$$\mathbb{E}(X|y) = \int_y^1 x\frac{2x}{1 - y^2}dx = \frac{2}{3}\frac{1 + y + y^2}{1 + y}$$