1. Exercise 3.1.1 on Page 146:

If the mgf of a random variable X is $\left(\frac{1}{3}+\frac{2}{3}e^t\right)^5$, find $P(X = 2\ or\ 3)$.

Answer:

Since the $M(t)$ of X is $\left(\frac{1}{3}+\frac{2}{3}e^t\right)^5$, X has a binomial distribution with $n = 5$, $p = \frac{2}{3}$.
The probability density function of the binomial distribution is

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & elsewhere \end{cases}$$

Thus,

$$\begin{aligned} P(X = 2\ or\ X = 3) &= P(X = 2) + P(X = 3) \\ &= \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 + \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 \\ &= \frac{40}{81} \end{aligned}$$

2. Exercise 3.1.27 on Page 149:

Consider a shipment of 1000 items into a factory. Suppose the factory can tolerate about 5% defective items. Let X be the number of defective items in a sample without replacement of size n = 10. Suppose the factory returns the shipment if $X \geq 2$.

(a) Obtain the probability that the factory returns a shipment of items which has 5% defective items.

(b) Suppose the shipment has 10% defective items. Obtain the probability that the factory returns such a shipment.

(c) Obtain approximations to the probabilities in parts (a) and (b) using appropriate binomial distributions.

Note : If you do not have access to a computer package with a hypergeometric command, obtain the answer to (c) only. This is what would have been done in practice 20 years ago. If you have access to R, then the command $dhyper(x, D, N-D, n)$ returns the probability in expression (3.1.7)

Answer:

The expression (3.1.7) is

$$p(x) = \frac{\binom{N-D}{n-x}\binom{D}{x}}{\binom{N}{n}}, \quad x = 0, 1, \ldots, n$$

In the following, let X be the number of defective items.

(a) From the given information, we have that $N = 1000$, $n = 10$, $D = 1000 \times 5\% = 50$

Since X has a hyper-geometric distribution, we have that

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \frac{\binom{1000-50}{10-0}\binom{50}{0}}{\binom{1000}{10}} - \frac{\binom{1000-50}{10-1}\binom{50}{1}}{\binom{1000}{10}} \\ &= 0.0853 \end{aligned}$$

(b) Since $D = 1000 \times 10\% = 100$, we have that

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \frac{\binom{1000-100}{10-0}\binom{100}{0}}{\binom{1000}{10}} - \frac{\binom{1000-100}{10-1}\binom{100}{1}}{\binom{1000}{10}} \\ &= 0.2637 \end{aligned}$$

(c) From the given information, when $n = 10$, $p = 0.05$, using the binomial distributions, we have that

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{10}{0} 0.05^0 (1 - 0.05)^{10-0} - \binom{10}{1} 0.05^1 (1 - 0.05)^{10-1} \\ &= 0.0861 \end{aligned}$$

When $n = 10$, $p = 0.1$, using the binomial distributions, we have that

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{10}{0} 0.1^0 (1 - 0.1)^{10-0} - \binom{10}{1} 0.1^1 (1 - 0.1)^{10-1} \\ &= 0.2639 \end{aligned}$$

3. Exercise 3.2.3 on Page 154:

In a lengthy manuscript, it is discovered that only 13.5 percent of the pages contain no typing errors. If we assume that the number of errors per page is a random variable with a Poisson distribution, find the percentage of pages that have exactly one error.

Answer:

Since the Poisson distribution is $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \ \ x = 0, 1, \cdots$

we have

$$P(X = 0) = \frac{e^{-\lambda}\lambda^0}{0!}$$

From the given information, $P(X = 0) = 13.5\%$. Therefore, we can obtain $\lambda = 2$.

Thus,

$$P(X = 1) = \frac{e^{-\lambda}\lambda^1}{1!} = \frac{2^1 e^{-2}}{1!} = 0.271.$$

4. Exercise 3.2.14 on Page 155:

Let X_1 and X_2 be two independent random variables. Suppose that X_1 and $Y = X_1 + X_2$ have Poisson distributions with means μ_1 and $\mu > \mu_1$, respectively. Find the distribution of X_2

Answer:

The mgfs of X_1 and Y that have Poisson distributions are given by, respectively

$$M_{X_1}(t) = exp\{(\mu_1(e^t - 1)\}$$

and

$$M_Y(t) = exp\{(\mu(e^t - 1)\}$$

Since X_1 and X_2 are independently random variables, we have

$$M_Y(t) = M_{X_1}(t)M_{X_2}(t)$$

and

$$exp\{(\mu(e^t - 1)\} = exp\{(\mu_1(e^t - 1)\}M_{X_2}(t)$$

Therefore,

$$M_{X_2}(t) = exp\{(\mu - \mu_1)(e^t - 1)\}$$

Thus, X_2 has poisson distribution with mean $(\mu - \mu_1)$.

5. Exercise 3.3.6 on Pages 164

Let X_1, X_2, and X_3 be iid random variables, each with pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere.
(a) Find the distribution of $Y = minimum(X_1, X_2, X_3)$.
$Hint: P(Y \leq y) = 1 - P(Y > y) = 1 - P(Xi > y, i = 1, 2, 3)$.
(b) Find the distribution of $Y = maximum(X_1, X_2, X_3)$.

Answer:
From the given information, we have that the cumulative density function is

$$F(x) = 1 - e^{-x}$$

(a)

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= 1 - P(Y > y) \\ &= 1 - P(X_1 > y, X_2 > y, X_3 > y) \\ &= 1 - [1 - P(X_1 \leq y)][1 - P(X_2 \leq y)][1 - P(X_3 \leq y)] \\ &= 1 - [1 - (1 - e^{-y})]^3 \\ &= 1 - e^{-3y} \end{aligned}$$

Thus the distribution of Y is

$$F_Y(y) = \begin{cases} 1 - e^{-3y} & y > 0 \\ 0 & elsewhere \end{cases}$$

(b)

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X_1 \leq y)P(X_2 \leq y)P(X_3 \leq y) \\ &= (1 - e^{-y})(1 - e^{-y})(1 - e^{-y}) \\ &= (1 - e^{-y})^3 \end{aligned}$$

Thus the distribution of Y is

$$F_Y(y) = \begin{cases} (1 - e^{-y})^3 & y > 0 \\ 0 & elsewhere \end{cases}$$

6. Exercise 3.4.15 on Page 176:

Let X be a random variable such that $E(X^{2m}) = (2m)!/(2^m m!), m = 1,2,3,\ldots$ and $E(X^{2m-1}) = 0, m = 1,2,3,\ldots$. Find the mgf and the pdf of X.

Answer:

The moment generating function of X is

$$\begin{aligned} M_X(t) &= Ee^{(tX)} \\ &= E\left[\sum_{n=0}^{\infty}\frac{(tX)^n}{n!}\right] \\ &= \sum_{n=0}^{\infty}\frac{E(X^n)t^n}{n!} \\ &= 1+\sum_{m=1}^{\infty}\frac{E(X^{2m})t^{2m}}{(2m)!} \quad (Since\ \ EX^{2m-1}=0) \\ &= \sum_{m=0}^{\infty}\frac{(t^2/2)^m}{m!} \\ &= e^{\frac{t^2}{2}} \end{aligned}$$

Thus, X follows $N(0,1)$, and the pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}}exp\left(-\frac{x^2}{2}\right), \quad -\infty < x < \infty$$

7. Exercise 3.6.12 on Page 196:

Show that

$$Y = \frac{1}{1+(r_1/r_2)W}$$

where W has an F -distribution with parameters r_1 and r_2, has a beta distribution.

Answer:

Let U and V are independent chi-square random variables with r_1 and r_2 degrees of freedom, respectively, and let $W = r_2U/(r_1V)$. Then we can obtain that W has an F-distribution with two parameters r_1 and r_2.

$$\begin{aligned} Y &= \frac{1}{1+(r_1/r_2)W} \\ &= \frac{1}{1+(r_1/r_2)(\frac{r_2U}{r_1V})} \\ &= \frac{V}{U+V} \end{aligned}$$

Since U and V have chi-square distribution with r_1 and r_2 degrees of freedom, we obtain that $\frac{V}{U+V}$ has a beta distribution. Thus Y has a beta distribution.

8. Exercise 3.6.14 on Page 196:

Let X_1, X_2, and X_3 be three independent chi-square variables with r_1, r_2, and r_3 degrees of freedom, respectively.

(a) Show that $Y_1 = X_1/X_2$ and $Y_2 = X_1 + X_2$ are independent and that Y_2 is $\chi^2(r_1 + r_2)$.

(b) Deduce that

$$\frac{X_1/r_1}{X_2/r_2} \quad and \quad \frac{X_3/r_3}{(X_1 + X_2)/(r_1 + r_2)}$$

are independent F-variables.

Answer:

(a)

Since $Y_1 = X_1/X_2$ and $Y_2 = X_1 + X_2$, we can obtain that $X_1 = Y_1Y_2/(1 + Y_1)$ and $X_2 = Y_2/(1 + Y_1)$. The determinant of Jacobian is

$$\begin{vmatrix} \frac{y_2}{(y_1+1)^2} & \frac{y_1}{y_1+1} \\ \frac{-y_2}{(y_1+1)^2} & \frac{1}{y_1+1} \end{vmatrix} = \frac{y_2}{(y_1 + 1)^2}$$

Therefore the joint pdf of X_1 and X_2 is

$$f(x_1, x_2) = \frac{1}{2^{\frac{r_1+r_2}{2}} \Gamma\left(\frac{r_1}{2}\right) \Gamma\left(\frac{r_2}{2}\right)} x_1^{\frac{r_1}{2}-1} x_2^{\frac{r_2}{2}-1} e^{-\frac{x_1+x_2}{2}}$$

The joint pdf of Y_1 and Y_2 is

$$\begin{aligned} f(y_1, y_2) &= \frac{1}{2^{\frac{r_1+r_2}{2}} \Gamma\left(\frac{r_1}{2}\right) \Gamma\left(\frac{r_2}{2}\right)} \left(\frac{y_1 y_2}{y_1 + 1}\right)^{\frac{r_1}{2}-1} \left(\frac{y_2}{y_1 + 1}\right)^{\frac{r_2}{2}-1} e^{-\frac{y_2}{2}} \frac{y_2}{(y_1 + 1)^2} \\ &= \frac{1}{2^{\frac{r_1+r_2}{2}} \Gamma\left(\frac{r_1}{2}\right) \Gamma\left(\frac{r_2}{2}\right)} \cdot \frac{y_1^{\frac{r_1}{2}-1} y_2^{\frac{r_1+r_2}{2}-1}}{(y_1 + 1)^{\frac{r_1+r_2}{2}}} \cdot e^{-\frac{y_2}{2}} \end{aligned}$$

Thus the joint pdf can be decomposed into two factors, each of them relevant to Y_1, Y_2 alone. Hence Y_1, Y_2 are independent.

Since X_1 and X_2 follow independently chi-square distribution with r_1, r_2, we have that Y has a chi-square distribution with $r_1 + r_2$.

(b)

Since X_1, X_2, and X_3 are three independent chi-square variables with r_1, r_2, and r_3 degrees of freedom, therefore

$$W = \frac{X_1/r_1}{X_2/r_2} \sim F(r_1, r_2)$$

$$Y_2 = X_1 + X_2 \sim \chi^2(r_1 + r_2)$$

From the known information, the ratio of two chi-square variables with their respective degrees of freedom follows F-distribution.

Let

$$\begin{aligned} V &= \frac{X_3/r_3}{Y_2/(r_1+r_2)} \\ &= \frac{X_3/r_3}{(X_1+X_2)/(r_1+r_2)} \sim F(r_3, (r_1+r_2)) \end{aligned}$$

X_1, X_2, and X_3 are independent, X_1/X_2, and Y_2 are independent. So W and V are independent. Thus

$$\frac{X_1/r_1}{X_2/r_2} \ and \ \frac{X_3/r_3}{(X_1+X_2)/(r_1+r_2)}$$

are independent F-variables.