1. Exercise 2.4.1 on Page 108:

Let the random variables X and Y have the joint p.m.f.

(a) $p(x, y) = \frac{1}{3}$, $(x, y) = (0, 0), (1, 1), (2, 2)$, zero elsewhere.

Answer:

$$\mu_1 = \mathbb{E}(X) = \sum_x x\frac{1}{3} = 1$$

$$\mu_2 = \mathbb{E}(Y) = \sum_y y\frac{1}{3} = 1$$

$$\mathbb{E}(XY) = \sum_x (xy)\frac{1}{3} = 1\frac{5}{3}$$

$$\sigma_1^2 = \sigma_2^2 = \mathbb{E}(X^2) - \mu_1^2 = \sum_x x^2 p(x) - 1 = \frac{2}{3}$$

$$\rho = \frac{\mathbb{E}[(X-\mu_1)(Y-\mu_2)]}{\sigma_1\sigma_2} = \frac{5/3 - 1^2}{2/3} = 1$$

(b)$p(x, y) = \frac{1}{3}$, $(x, y) = (0, 2), (1, 1), (2, 0)$, zero elsewhere.

Answer:

$$\mu_1 = \mathbb{E}(X) = \sum_x x\frac{1}{3} = 1$$

$$\mu_2 = \mathbb{E}(Y) = \sum_y y\frac{1}{3} = 1$$

$$\mathbb{E}(XY) = \sum_x (xy)\frac{1}{3} = \frac{1}{3}$$

$$\sigma_1^2 = \sigma_2^2 = \mathbb{E}(X^2) - \mu_1^2 = \sum_x x^2 p(x) - 1 = \frac{2}{3}$$

$$\rho = \frac{\mathbb{E}[(X-\mu_1)(Y-\mu_2)]}{\sigma_1\sigma_2} = \frac{1/3 - 1^2}{2/3} = -1$$

(c)$p(x, y) = \frac{1}{3}$, $(x, y) = (0, 0), (1, 1), (2, 0)$, zero elsewhere.

Answer:

$$\mu_1 = \mathbb{E}(X) = \sum_x x\frac{1}{3} = 1$$

$$\mu_2 = \mathbb{E}(Y) = \sum_y y\frac{1}{3} = 1/3$$

$$\mathbb{E}(XY) = \sum_x (xy)\frac{1}{3} = \frac{1}{3} = \mu_1\mu_2$$

$$\rho = \frac{\mathbb{E}[(X-\mu_1)(Y-\mu_2)]}{\sigma_1\sigma_2} = 0$$

2. Exercise 2.4.4 on Page 109

Show that the variance of the conditional distribution of Y, Given X=x, in Exercise 2.4.3, is $\frac{(1-x)^2}{12}$, $0 < x < 1$, and that the variance of the conditional distribution of X, given $Y = y$, is $\frac{y^2}{12}$, $0 < y < 1$.

Answer:

For 2.4.3, we have

$$f(x,y) = 2, \qquad 0 < x < y, \qquad 0 < y < 1$$

Then

$$f(x) = \int_x^1 f(x,y)dy = 2(1-x) \qquad 0 < x < 1$$
$$f(y) = \int_0^y f(x,y)dx = 2y \qquad 0 < x < 1$$

Then,

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{1}{1-x} \qquad x < y < 1$$
$$\mathbb{E}(y|x) = \int_x^1 y\frac{1}{1-x}dy = \frac{1+x}{2}$$
$$\mathbb{E}(y^2|x) = \int_x^1 y2\frac{1}{1-x}dy = \frac{1+x+x^2}{3}$$
$$\mathbb{V}(y|x) = \frac{1+x+x^2}{3} - (\frac{1+x}{2})^2 = \frac{(1-x)^2}{12}$$

Meanwhile,

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{1}{y} \qquad 0 < x < y$$
$$\mathbb{E}(x|y) = \int_0^y x\frac{1}{y}dx = \frac{1}{2}y$$
$$\mathbb{E}(x^2|y) = \int_0^y x^2\frac{1}{y}dx = \frac{1}{3}y^2$$
$$\mathbb{V}(x|y) = \frac{1}{3}y^2 - \frac{1}{4}y^2 = \frac{y^2}{12}$$

3. Exercise 2.4.7 on Page 109

If the correlation coefficient ρ of X and Y exists, show that$-1 \leq \rho \leq 1$. Hint consider the discriminant of the non-negative quadratic function:

$$h(v) = \mathbb{E}\{[(X-\mu_1) + v(Y-\mu_2)]^2\}$$

where v is real and is not a function of X nor of Y.

Answer:

$$h(v) = Var(X) + 2vCov(X,Y) + v^2Var(Y) \geq 0$$

Then, the discriminant of this quadratic must satisfy $b^2 - 4ac \leq 0$, which yields

$$[2Cov(X,Y)]^2 - 4Var(X)Var(Y) \leq 0$$

Equivalently,

$$\rho^2 = \frac{Cov(X,Y)^2}{Var(X)Var(Y)} \leq 1$$

4. Exercise 2.4.10 on Page 109

Let X_1 and X_2 have the joint p.m.f described by the following table:

(x_1, x_2)	(0,0)	(0,1)	(0,2)	(1,1)	(1,2)	(2,2)
$p(x_1, x_2)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{1}{12}$

Find $p_1(x_1)$, $p_2(x_2)$, μ_1, μ_2, σ_1^2, σ_2^2, and ρ.

Answer:

We have

x_1	0	1	2	Total
$p(x_1)$	$\frac{4}{12}$	$\frac{7}{12}$	$\frac{1}{12}$	1

x_2	0	1	2	Total
$p(x_2)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{6}{12}$	1

Then, we have the

$$\mu_1 = \sum_{x_1} x_1 p(x_1) = \frac{9}{12} = \frac{3}{4}$$

$$\mathbb{E}(x_1^2) = \sum_{x_1} x_1^2 p(x_1) = \frac{11}{12}$$

$$\mathbb{V}(x_1) = \frac{11}{12} - (\frac{3}{4})^2 = 0.3541667$$

$$\mu_2 = \sum_{x_2} x_2 p(x_2) = \frac{17}{12}$$

$$\mathbb{E}(x_2^2) = \sum_{x_2} x_2^2 p(x_2) = \frac{29}{12}$$

$$\mathbb{V}(x_2) = \frac{29}{12} - (\frac{17}{12})^2 = 0.4097222$$

Then the ρ, we have

$$\rho = \frac{E(x_1x_2) - E(x_1)E(x_2)}{SD(x_1)SD(x_2)} = \frac{(3+8+4)/12 - 3/4 \times 17/12}{0.380933} = 0.4922125$$

5. Exercise 2.5.2 on Page 116

If the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = 2e^{-x_1-x_2}$, $0 < x_1 < x_2$, $0 < x_2 < \infty$, zero elsewhere, show that X_1 and X_2 are dependent.
Answer:

$$f(x_1) = \int_{x_1}^{\infty} 2e^{-x_1-x_2} dx_2 = 2e^{-2x_1}$$
$$f(x_2) = \int_{0}^{x_2} 2e^{-x_1-x_2} dx_2 = 2e^{-x_2} - 2e^{-2x_2}$$

Therefore,

$$f(x_1, x_2) \neq f(x_1)f(x_2)$$

Hence, they are dependent.

6. Exercise 2.5.5 on Page 116

Find the probability of the union of the events $a < X_1 < b$, $-\infty < X_2 < \infty$, and $-\infty < X_1 < \infty$, $c < X_2 < d$ if X_1 and X_2 are two independent variables with $P(a < X_1 < b) = \frac{2}{3}$ and $P(c < X_2 < d) = \frac{5}{8}$.
Answer:

$$\frac{2}{3} + \frac{5}{8} - \frac{2}{3} \times \frac{5}{8} = \frac{7}{8}$$

7. Exercise 2.5.13 on Page 117

For X_1 and X_2 in Example 2.5.6, show that the m.g.f of $Y = X_1 + X_2$ is $\frac{e^{2t}}{(2-e^t)^2}$, $t < \log(2)$, and then compute the mean and variance of Y.
Answer:
As X_1 and X_2 are independent random variable with m.g.f:

$$\begin{aligned} M_{X_1}(t) = M_{X_2}(t) &= \sum_{2}^{\infty} e^{tx} (\frac{1}{2})^x \\ &= \sum_{1}^{\infty} (\frac{1}{2}e^t)^x \\ &= (\frac{e^t}{2}) \frac{1}{1 - \frac{e^t}{2}} \\ &= \frac{e^t}{2 - e^t} \end{aligned}$$

Then

$$M_Y(t) = M_{X_1}(t) \times M_{X_2}(t) = \frac{e^{2t}}{(2 - e^t)^2}$$