1. Exercise 3.1.2 on Page 146:

The mgf of a random variable X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$. Show that

$$P(\mu-2\sigma < X < \mu+2\sigma) = \sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}.$$

Answer:

Since the $M(t)$ of X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$, we have that X is a binomial distribution with $n=9$, $p=\frac{1}{3}$, $\mu=np=3$, $\sigma^2=np(1-p)=2$. Therefore, $\mu-2\sigma=3-2\sqrt{2}$,$\mu+2\sigma=3+2\sqrt{2}$. Thus

$$P(\mu-2\sigma < X < \mu+2\sigma) = P(X=1,2,3,4,5) = \sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}.$$

2. Exercise 3.1.16 on Page 148:

Show that the moment generating function of the negative binomial distribution is $M(t) = p^r[1-(1-p)e^t]^{-r}$. Find the mean and the variance of this distribution.

Hint : In the summation representing $M(t)$, make use of the Maclaurin's series for $(1-w)^{-r}$.

Answer:

Let Y has a negative binomial distribution, so we have that

$$P(Y=y) = \binom{y+r-1}{r-1}p^r(1-p)^y, \quad y=0,1,2\cdots$$

$$\begin{aligned} M(t) &= Ee^{tY} \\ &= \sum_{y=0}^{\infty}\binom{y+r-1}{r-1}e^{ty}p^r(1-p)^y, \quad y=0,1,2\cdots \\ &= p^r\sum_{y=0}^{\infty}\binom{y+r-1}{y}[(1-p)e^t]^y, \quad y=0,1,2\cdots \\ &= p^r[1-(1-p)e^t]^{-r} \end{aligned}$$

Because

$$(1-w)^{-r} = \sum_{y=0}^{\infty}\binom{y+r-1}{y}w^y,$$

where $w=(1-p)e^t$.

$$E(Y) = \frac{r(1-p)}{p}, \quad Var(Y) = \frac{r(1-p)}{p^2}$$

3. Exercise 3.2.1 on Page 154:

If the random variable X has a Poisson distribution such that $P(X = 1) = P(X = 2)$, find $P(X = 4)$.

Answer:

Since $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, so

$$\frac{e^{-\lambda}\lambda}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

Then, we have that $\lambda = 2$.

Thus,

$$P(X = 4) = \frac{2^4 e^{-2}}{4!} = \frac{2}{3}e^{-2}.$$

4. Exercise 3.2.13 on Page 155:

Let X and Y have the joint pmf $p(x, y) = e^{-2}/[x!(y-x)!]$, $y = 0, 1, 2, \ldots,\ x = 0, 1, \ldots, y$, zero elsewhere.

(a) Find the mgf $M(t_1, t_2)$ of this joint distribution.

(b) Compute the means, the variances, and the correlation coefficient of X and Y.

(c) Determine the conditional mean $E(X|y)$.

Hint : Note that

$$\sum_{x=0}^{y}[exp(t_1 x)]y!/[x!(y-x)!] = [1 + exp(t_1)]^y.$$

Answer: for (a),

$$\begin{aligned}
M(t_1, t_2) &= Ee^{(t_1X+t_2Y)} \\
&= \sum_{y=0}^{\infty}\sum_{x=0}^{y} e^{(t_1x+t_2y)}\frac{e^{-2}}{x!(y-x)!} \\
&= \sum_{y=0}^{\infty}\frac{e^{t_2y}e^{-2}}{y!}\sum_{x=0}^{y}\frac{e^{t_1x}y!}{x!(y-x)!} \\
&= \sum_{y=0}^{\infty}\frac{e^{t_2y}e^{-2}}{y!}(1+e^{t_1})^y \\
&= \sum_{y=0}^{\infty}\frac{[e^{t_2}(1+e^{t_1})]^y e^{-2}}{y!} \\
&= e^{-2}exp\{(1+e^{t_1})e^{t_2}\}.
\end{aligned}$$

For (b),
$E(X) = 1,\ E(Y) = 2,\ E(X^2) = 2,\ E(Y^2) = 6,\ Var(X) = 1,\ Var(Y) = 2,\ \rho = \frac{1}{\sqrt{2}}$.
For (c),

$$P(Y = y) = \sum_{x=0}^{y} \frac{e^{-2}}{x!(y-x)!} = \frac{e^{-2}}{y!} \sum_{x=0}^{y} \frac{y!}{x!(y-x)!} = \frac{2^y e^{-2}}{y!}$$

$$P(X|Y = y) = \frac{e^{-2}/[x!(y-x)!]}{(2^y e^{-2})/y!} = \frac{y!}{x!(y-x)!} \cdot \frac{1}{2^y}$$

$$E(X|Y = y) = \sum_{x=0}^{y} x \cdot \frac{y!}{x!(y-x)!} \cdot \frac{1}{2^y} = \frac{y}{2^y} \left[\sum_{x=1}^{y} \cdot \frac{(y-1)!}{(x-1)!(y-x)!} \right] = \frac{y}{2}.$$

5. Exercise 3.3.24 on Page 166:

Let X_1, X_2 be two independent random variables having gamma distributions with parameters $\alpha_1 = 3, \beta_1 = 3$ and $\alpha_2 = 5, \beta_2 = 1$, respectively.
(a) Find the mgf of $Y = 2X_1 + 6X_2$.
(b) What is the distribution of Y ?

Answer:

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty.$$

For (a), since

$$\begin{aligned}
Ee^{2tX} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{2tx} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{1}{\beta^\alpha} (\frac{\beta y}{1-2t\beta})^{\alpha-1} e^{-x/\beta} (\frac{\beta}{1-2t\beta}) dy, \quad \left(Let\ y = \frac{x(1-2t\beta)}{\beta} \right) \\
&= \frac{1}{(1-2t\beta)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy \\
&= \frac{1}{(1-2t\beta)^\alpha}
\end{aligned}$$

Similarly, we obtain that

$$Ee^{6tX} = \frac{1}{(1-6t\beta)^\alpha}$$

So we have that

$$M(t) = Ee^{t(2X_1+6X_2)} = Ee^{2tX_1} \cdot Ee^{6tX_2} = (1-6t)^{-8}, \ (t < 1/6).$$

For (b), by the mgf of Y in (a), we have that Y has a Gamma distribution and $\alpha = 8,\ \beta = 6$.

6. Exercise 3.4.28 on Page 178:

Let X_1 and X_2 be independent with normal distributions $N(6,1)$ and $N(7,1)$, respectively. Find $P(X_1 > X_2)$.
Hint : Write $P(X_1 > X_2) = P(X_1 - X_2 > 0)$ and determine the distribution of $X_1 - X_2$.

Answer: Let Y=$X_1 - X_2$, since X_1 and X_2 are independent with normal distributions, we have that

$$M_{X_1}(t) = Ee^{tX_1} = e^{(6t+\frac{1}{2}t^2)},\ M_{(-X_2)}(t) = Ee^{tX_2} = e^{(-7t+\frac{1}{2}t^2)}.$$

and

$$M_Y(t) = Ee^{tY} = Ee^{tX_1}Ee^{-tX_2} = e^{(-t+t^2)}.$$

Therefore, we obtain that Y has a normal distribution $N(-1,2)$. Thus,

$$P(X_1 - X_2 > 0) = P(Y > 0) = 1 - \Phi(1/\sqrt{2}).$$

7. Exercise 3.6.10 on Page 196:

Let $T = W/\sqrt{V/r}$, where the independent variables W and V are, respectively, normal with mean zero and variance 1 and chi-square with r degrees of freedom. Show that T^2 has an F-distribution with parameters $r_1 = 1$ and $r_2 = r$.
Hint : What is the distribution of the numerator of T^2?

Answer:
Since $T = W/\sqrt{V/r}$, we have that $T^2 = W^2/(V/r) = (W^2/1)/(V/r)$. Moreover, W is $N(0,1)$, then we have that W^2 is chi-square with 1 degrees of freedom. The variables W and V are independent, so W^2 and V are independent. Thus, T^2 is F-distribution with 1 and r degrees of freedom.

8. Exercise 3.6.13 on Page 196:

Let X_1, X_2 be iid with common distribution having the pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere. Show that $Z = X_1/X_2$ has an F-distribution.

Answer:
This question I explained wrongly in the class, as you can not multiply the m.g.f. directly

as X_1 and X_1 itself are dependent.

Therefore, we need to use the Jacobian:

$$|\frac{dx}{dy}| = \frac{1}{2}$$

Then, the p.d.f. is

$$g(y) = f(\frac{1}{2}y) \times \frac{1}{2} = \frac{1}{2}e^{-\frac{y}{2}}$$

Then correspondingly we can have the m.g.f.

$$M_{Y_i}(t) = (1 - 2t)^{-2}, \ \ (t < 1/2).$$

Or otherwise you do not use the m.g.f, just directly observe the p.d.f and recognise that Y_i is chi-square with 2 degrees of freedom. Moreover, X_1 and X_2 are iid, so we have that

$$Z = \frac{X_1}{X_2} = \frac{2X_1/2}{2X_2/2} = \frac{Y_1}{Y_2}$$

Thus Z has an F-distribution.