1. Exercise 4.5.4 on Page 246

Let X have a binomial distribution with the number of trials $n = 10$ and with p either $1/4$ or $1/2$. The simple hypothesis $H_0 : p = 1/2$ is rejected, and the alternative simple hypothesis $H_1 : p = 1/4$ is accepted, if the observed value of X_1, a random sample of size 1, is less than or equal to 3. Find the significance level and the power of the test.
Answer:

$$\alpha = Pr(X_1 \leq 3|H_0) = \sum_{i=1}^{3} \binom{10}{i} (1/2)^i (1/2)^{10-i} = 0.17 \tag{1}$$

$$1 - \beta = Pr(X_1 \leq 3|H_1) = \sum_{i=1}^{3} \binom{10}{i} (1/4)^i (3/4)^{10-i} = 0.7758751 \tag{2}$$

$$\alpha = 0.17 \qquad 1 - \beta = 0.78 \tag{3}$$

2. Exercise 4.6.6 on Page 253

Each of 51 golfers hit three golf balls of brand X and three golf balls of brand Y in a random order. Let X_i and Y_i equal the averages of the distances traveled by the brand X and brand Y golf balls hit by the i-th golfer, $i = 1, 2, ..., 51$. Let $W_i = X_i - Y_i$, $i = 1, 2, ..., 51$. Test $H_0 : \mu_W = 0$ against $H_1 : \mu_W > 0$, where μ_W is the mean of the differences. If $w = 2.07$ and $s_W^2 = 84.63$, would H_0 be accepted or rejected at an $\alpha = 0.05$ significance level? What is the p-value of this test?
Answer:

$$t = \frac{2.07 - 0}{\sqrt{84.63/51}} = 1.606916 \sim t(50) \tag{4}$$

$$\text{p-value} = 1 - 0.9428148 = 0.05718519 > 0.05$$

Also

$$t^*(50, 0.95) = 1.675905 > t$$

Therefore, we do not reject H_0.

3. Exercise 4.7.3 on Page 260

A die was cast $n = 120$ independent times and the following data resulted:

Spots Up	1	2	3	4	5	6
Frequency	b	20	20	20	20	$40 - b$

If we use a chi-square test, for what values of b would the hypothesis that the die is unbiased be rejected at the 0.025 significance level?

Answer:

Using χ^2 test, we have

$$\sum_{i=1}^{6}\frac{(X_i-np_i)^2}{np_i}=\frac{(b-20)^2}{20}+\frac{(40-b-20)^2}{20}>\chi^2(5,0.975)=12.8325 \tag{5}$$

Therefore,

$$b<8.7 \quad \text{or} \quad b>31.3$$

4. Exercise 5.3.1 on Page 313

Let X denote the mean of a random sample of size 100 from a distribution that is $\chi^2(50)$. Compute an approximate value of $P(49<X<51)$.

Answer:

For $\chi^2(50)$,

$$\mathbb{E}(X)=50 \qquad \mathbb{V}(X)=100 \tag{6}$$

Hence, by CLT,

$$\begin{aligned} Pr(49<\bar{X}<51)&=Pr(\sqrt{n}\frac{49-\mu}{\sigma}<Z<\sqrt{n}\frac{51-\mu}{\sigma})\\ &=\Phi(1)-\Phi(-1)\\ &=0.6827 \end{aligned} \tag{7}$$

5. Exercise 6.1.1 on Page 326

Let $X_1,X_2,...,X_n$ be a random sample from a $\Gamma(\alpha=3,\beta=\theta)$ distribution, $0<\theta<\infty$. Determine the mle of θ.

Answer:

For the MLE of Gamma distribution,

$$\begin{aligned} L(\beta;\alpha,x_i)&=\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x_i^{\alpha-1}e^{-x_i/\beta}\\ \ell(\beta;\alpha,x_i)&=-n\log(\Gamma(\alpha))-n\alpha\log(\beta)+(\alpha-1)\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}x_i/\beta\\ \frac{\partial\ell(\beta;\alpha,x_i)}{\partial\beta}&=\frac{\sum_{i=1}^{n}x_i}{\beta^2}-\frac{n\alpha}{\beta}=0\\ \Rightarrow \qquad \hat{\beta}&=\frac{\sum_{i=1}^{n}x_i}{n\alpha}=\frac{\sum_{i=1}^{n}x_i}{3n} \end{aligned} \tag{8}$$

6. Exercise 6.2.7 on Page 340

Let X have a gamma distribution with $\alpha = 4$ and $\beta = \theta > 0$.

(a) Find the Fisher information $I(\theta)$.

(b) If $X_1, X_2, ..., X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ.

(c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

Answer:

From Question 5, we have

$$\begin{aligned}
&\log f(x;\beta) = -\log(\Gamma(\alpha)) - \alpha\log(\beta) + (\alpha-1)\log(x) - x/\beta \\
&\frac{\partial \log f(x;\beta)}{\partial \beta} = -\frac{\alpha}{\beta} + \frac{x}{\beta^2} \\
&\frac{\partial^2 \log f(x;\beta)}{\partial \beta^2} = \frac{\alpha}{\beta^2} - 2\frac{x}{\beta^3} \\
&I(\beta) = -\mathbb{E}(\frac{\partial^2 \log f(x;\beta)}{\partial \beta^2}) = -\frac{\alpha}{\beta^2} + 2\frac{\mathbb{E}(X)}{\beta^3} = \frac{\alpha}{\beta^2}
\end{aligned} \tag{9}$$

$$\text{where} \quad \mathbb{E}(X) = \alpha\beta$$

(b)As from question 5

$$Var(\hat{\beta}) = \frac{Var(X)}{n\alpha^2} = \frac{\alpha\beta^2}{n\alpha^2} = \frac{\beta^2}{n\alpha} = \frac{1}{nI(\beta)} \tag{10}$$

Therefore, mle is an efficient estimator.

(c)As

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, \frac{1}{I(\theta)})$$

then,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, \theta^2/\alpha)$$