1. Exercise 1.9.1 on Page 64

Find the mean and variance, if they exist, of each of the following distribution.
(a) $p(x) = \frac{3!}{x!(3-x)!}(\frac{1}{2})^3$, $x = 0, 1, 2, 3$, zero elsewhere.
Answer(a):

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{x=0}^{3} xp(x) = \sum_{x=0}^{3} x\frac{3!}{x!(3-x)!}(\frac{1}{2})^3 \\
&= \sum_{x=1}^{3} \frac{2!}{(x-1)!(2-(x-1))!} \times (\frac{1}{2})^2 \times 3 \times \frac{1}{2} \\
&= [\sum_{x=0}^{2} \frac{2!}{(y)!(2-(y))!} \times (\frac{1}{2})^2] \times 3 \times \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}(X(X-1)+X) \\
&= \sum_{x=0}^{3} x(x-1)p(x) + E(X) \\
&= \sum_{x=0}^{3} x(x-1)\frac{3!}{x!(3-x)!}(\frac{1}{2})^3 + E(X) \\
&= \sum_{x=2}^{3} \frac{1!}{(x-2)!(1-(x-2))!} \times \frac{1}{2} \times 3 \times 2 \times (\frac{1}{2})^2 + E(X) \\
&= [\sum_{y=0}^{1} \frac{1!}{(y)!(1-y)!} \times \frac{1}{2}] \times 3 \times 2 \times (\frac{1}{2})^2 + E(X) \\
&= 3 \times 2 \times (\frac{1}{2})^2 + \frac{3}{2} \\
&= 3
\end{aligned}$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3 - (\frac{3}{2})^2 = \frac{3}{4}$$

This is a binomial distribution with $n = 3, p = \frac{1}{2}$, with known knowledge $\mathbb{E}(X) = np = \frac{3}{2}$, $\mathbb{V}(X) = np(1-p) = \frac{3}{4}$. It is important to know how to prove the equation in general. The general idea has been demonstrated. You may try to prove the n and p case yourself.

(b)$f(x) = 6x(1-x)$, $0 < x < 1$, zero elsewhere.
Answer(b):

$$\mathbb{E}(X) = \int_0^1 xf(x)dx = \int_0^1 6x^2 - 6x^3)dx = 2x^3 - \frac{3}{2}x^4|_0^1 = \frac{1}{2}$$

$$\mathbb{E}(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 6x^3 - 6x^4 dx = \frac{3}{2}x^4 - \frac{6}{5}x^5|_0^1 = \frac{3}{10}$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{20}$$

(c)$f(x) = \frac{2}{x^3}$, $1 < x < \infty$, zero elsewhere.

Answer(c):

$$\mathbb{E}(X) = \int_1^\infty x\frac{2}{x^3}dx = -\frac{2}{x}|_1^\infty = 2$$

$$\mathbb{E}(X^2) = \int_1^\infty \frac{2}{x}dx = 2\ln(x)|_1^\infty \to \infty$$

This is a distribution with finite mean and infinite variance. You may notice that there is slight difference between the variance does not exist and infinite variance.

(Note: Though this question is not asked, but you can check this example does not have moment generating function and the second moment is greater than the harmonic series.)

(You may check Example 1.9.4 for how to prove m.g.f does not exist, very similar.)

2. Exercise 1.9.7 on Page 65

Show that the moment generating function of the random variable X having the p.d.f $f(x) = \frac{1}{3}$, $-1 < x < 2$ zero elsewhere is

$$M(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t} & t \neq 0 \\ 1 \quad t = 0 \end{cases}$$

Answer:

When $t \neq 0$

$$\begin{aligned} M(t) &= \int_{-1}^2 e^{xt}\frac{1}{3}dx \\ &= \frac{1}{3t}e^{xt}|_{-1}^2 \\ &= \frac{e^{2t} - e^{-t}}{3t} \end{aligned}$$

When $t = 0$

$$M(t) = \int_{-1}^2 \frac{1}{3}dx = 1$$

3. Exercise 1.9.15 on Page 66

Let X be a random variable with mean μ and variance σ^2 such that the fourth moment $\mathbb{E}[(X-\mu)^4]$ exists. The value of the ratio $\frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4}$ is often used as a measure of *kurtosis*. Graph each of the following probability density functions and show that this measure is smaller for the first distribution.

(a) $f(x) = \frac{1}{2}$, $-1 < x < 1$, zero elsewhere.

(b) $f(x) = 3(1-x^2)/4$, $-1 < x < 1$, zero elsewhere.

Answer:

For (a):

$$\mathbb{E}(X) = \int_{-1}^{1} \frac{1}{2} x dx = \frac{1}{4} x^2 |_{-1}^{1} = 0$$

$$\sigma^2 = \mathbb{E}[(X-\mu)^2] = \int_{-1}^{1} \frac{1}{2} x^2 dx = \frac{1}{6} x^3 |_{-1}^{1} = \frac{1}{3}$$

$$\mathbb{E}[(X-\mu)^4] = \mathbb{E}(X^4) = \int_{-1}^{1} \frac{1}{2} x^4 dx = \frac{1}{10} x^5 |_{-1}^{1} = \frac{1}{5}$$

$$\frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4} = \frac{9}{5}$$

For (b):

$$\mathbb{E}(X) = \int_{-1}^{1} (\frac{3}{4}x - \frac{3}{4}x^3) dx = (\frac{3}{8}x^2 - \frac{3}{16}x^4)|_{-1}^{1} = 0$$

$$\sigma^2 = \mathbb{E}[(X-\mu)^2] = \int_{-1}^{1} (\frac{3}{4}x^2 - \frac{3}{4}x^4) dx = (\frac{1}{4}x^3 - \frac{3}{20}x^5)|_{-1}^{1} = \frac{1}{5}$$

$$\mathbb{E}[(X-\mu)^4] = \mathbb{E}(X^4) == \int_{-1}^{1} (\frac{3}{4}x^4 - \frac{3}{4}x^6) dx = (\frac{3}{20}x^5 - \frac{3}{28}x^7)|_{-1}^{1} = \frac{3}{35}$$

$$\frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4} = \frac{3}{35} \times 25 = \frac{15}{7} > \frac{9}{5}$$

4. Exercise 1.9.24 on Page 67

Consider k continuous-type distribution with the following characteristics: p.d.f. $f_i(x)$, mean μ_i, and variance σ_i^2, $i = 1, 2, ..., k$. If $c_i \geq 0$, $i = 1, 2, ..., k$, and $c_1 + c_2 + ... + c_k = 1$, show that the mean and variance of the distribution having p.d.f. $c_1 f_1(x) + c_2 f_2(x) + ... + c_k f_k(x)$ are $\mu = \sum_{i=1}^{k} c_i \mu_i$ and $\sigma^2 = \sum_{i=1}^{k} c_i [\sigma_i^2 + (\mu_i - \mu)^2]$, respectively.

Answer:

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(c_1 X_1 + c_2 X_2 + ... + c_k X_K) \\
&= \mathbb{E}(c_1 X_1) + ...\mathbb{E}(c_k X_k) \\
&= c_1 \mathbb{E}(X_1) + ... + c_k \mathbb{E}(X_k) \\
&= \sum_{i=1}^{k} c_i \mu_i
\end{aligned}$$

As

$$\begin{aligned}
\int_{-\infty}^{\infty} (x-\mu)^2 f_i(x) dx &= \int_{-\infty}^{\infty} (x - \mu_i + \mu_i - \mu)^2 f_i(x) dx \\
&= \int_{-\infty}^{\infty} (x-\mu_i)^2 f_i(x) dx + \int_{-\infty}^{\infty} 2(x-\mu_i)(\mu_i - \mu) f_i(x) dx + \int_{-\infty}^{\infty} (\mu - \mu_i)^2 f_i(x) dx \\
&= \sigma_i^2 + 0 + (\mu_i - \mu)^2
\end{aligned}$$

$$\mathbb{E}[(X-\mu)^2] = \sum_{i=1}^{k} c_i (\int_{-\infty}^{\infty} (x-\mu)^2 f_i(x) dx) = \sum_{i=1}^{k} c_i [\sigma_i^2 + (\mu_i - \mu)^2]$$

5. Exercise 2.1.9 on Page 83

Let the random variables X_1 and X_2 have the joint p.m.f. described as follows:

(x_1, x_2)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$p(x_1, x_2)$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

and $p(x_1, x_2)$ is equal to zero elsewhere.

(a) Write these probabilities in rectangular array as in Example 2.1.3, recording each marginal p.d.f in the 'margins'.

Answer(a):

		x_2		
x_1	0	1	2	$p_1(x_1)$
0	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{7}{12}$
1	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{5}{12}$
$p_2(x_2)$	$\frac{4}{12}$	$\frac{5}{12}$	$\frac{3}{12}$	

(b) What is $P(X_1 + X_2 = 1)$?

$$P(X_1 + X_2 = 1) = P(X_1 = 1, X_2 = 0) + P(X_1 = 0, X_2 = 0) = \frac{2}{12} + \frac{3}{12} = \frac{5}{12}$$

6. Exercise 2.1.12 on Page 84

Let X_1, X_2 be two random variables with the joint p.m.f. $p(x_1, x_2) = (x_1 + x_2)/12$, for $x_1 = 1, 2$, $x_2 = 1, 2$, zero elsewhere. Compute $\mathbb{E}(X_1)$, $\mathbb{E}(X_1^2)$, $\mathbb{E}(X_2)$, $\mathbb{E}(X_2^2)$, and $\mathbb{E}(X_1X_2)$. Is $\mathbb{E}(X_1X_2) = \mathbb{E}(X_1)\mathbb{E}(X_2)$? Find $\mathbb{E}(2X_2 - 6X_2^2 + 7X_1X_2)$.

Answer:

$$p(x_1) = \sum_{x_2=1,2} \frac{x_1 + x_2}{12} = \frac{2x_1 + 3}{12} \qquad x_1 = 1, 2$$

$$p(x_2) = \sum_{x_1=1,2} \frac{x_1 + x_2}{12} = \frac{2x_2 + 3}{12} \qquad x_2 = 1, 2$$

Therefore, as X_1, X_2 are symmetric, they have the same p.m.f. and same expectations.

$$\mathbb{E}(X_1) = \sum_{x_1=1,2} x_1 \frac{2x_1 + 3}{12} = \frac{2 + 3 + 8 + 6}{12} = \frac{19}{12}$$

$$\mathbb{E}(X_2) = \mathbb{E}(X_1) = \frac{19}{12}$$

Similarly, we have

$$\mathbb{E}(X_1^2) = \mathbb{E}(X_2^2) = \sum_{x_1} x_1^2 \frac{2x_1+3}{12} = \frac{2+3+16+12}{12} = \frac{33}{12}$$

$$\mathbb{E}(X_1X_2) = \sum_{x_1,x_2} x_1x_2\frac{x_1+x_2}{12} = 1\times 1\frac{1+1}{12} + 1\times 2\frac{1+2}{12} + 2\times 1\frac{2+1}{12} + 2\times 2\frac{2+2}{12} = \frac{2+6+6+16}{12}$$
$$= \frac{30}{12} = \frac{5}{2} \neq \mathbb{E}(X_1)\mathbb{E}(X_2)$$

Therefore,

$$\mathbb{E}(2X_2 - 6X_2^2 + 7X_1X_2) = 2\mathbb{E}(X_2) - 6\mathbb{E}(X_2^2) + 7\mathbb{E}(X_1X_2) = 2\times\frac{19}{12} + 6\times\frac{33}{12} + 7\times\frac{5}{2} = \frac{223}{6}$$

7. Exercise 2.2.1 on Page 92

If $p(x_1,x_2) = (\frac{2}{3})^{x_1+x_2}(\frac{1}{3})^{2-x_1-x_2}$, $(x_1,x_2) = (0,0),(0,1),(1,0),(1,1)$, zero elsewhere, is the joint p.m.f. of X_1 and X_2, find the joint p.m.f. of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

Answer:

$$p(y_1,y_2) = \begin{cases} (\frac{2}{3})^{y_2}(\frac{1}{3})^{2-y_2} & \text{when } (y_1,y_2) = (0,0),(-1,1),(1,1),(0,2) \\ 0 & \end{cases}$$

This question is trivial, you can just use the substitution directly, otherwise you need to use the Jacobian for some more complex cases.

8. Exercise 2.3.2 on Page 100

Let $f_{1|2}(x_1|x_2) = c_1\frac{x_1}{x_2^2}$, $0 < x_1 < x_2$, $0 < x_1 < 1$, zero elsewhere, and $f_2(x_2) = c_2x_2^4$, $0 < x_2 < 1$, zero elsewhere, denote, respectively, the conditional p.d.f. of X_1, given $X_2 = x_2$, and the marginal p.d.f. of X_2. Determine:

(a) The constants of c_1 and c_2:

Answer(a):

$$c_1\int_0^{x_2}\frac{x_1}{x_2^2}dx_1 = \frac{c_1}{2} = 1 \qquad \rightarrow \qquad c_1 = 2$$

Similarly,

$$\int_0^1 c_2x_2^4dx_2 = 1 \qquad \rightarrow \qquad c_2 = 5$$

(b) The joint p.d.f. of X_1 and X_2.

Answer(b):

The joint pdf is

$$f(x_1,x_2) = f_{1|2}(x_1|x_2)\times f_2(x_2) = 10x_1x_2^2, \qquad 0 < x_1 < x_2 < 1$$

(c) $P(\frac{1}{4} < X_1 < \frac{1}{2}|X_2 = \frac{5}{8})$.

Answer(c):

$$P(\frac{1}{4} < X_1 < \frac{1}{2}|X_2 = \frac{5}{8}) = \int_{\frac{1}{4}}^{\frac{1}{2}} 2x_1/(\frac{5}{8})^2 dx = \frac{12}{25}$$

(d) $P(\frac{1}{4} < X_1 < \frac{1}{2})$.

Answer(d):

$$P(\frac{1}{4} < X_1 < \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{1}^{x_1} 10x_1x_2^2 dx_2 x_1 = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{10}{3} x_1(1 - x_1^3) dx_1 = \frac{135}{512}$$

9. Exercise 2.3.8 on Page 101

Let X and Y have the joint p.d.f $f(x, y) = 2\exp(-(x + y))$, $0 < x < y < \infty$, zero elsewhere. Find the conditional mean $\mathbb{E}(Y|x)$ of Y, given $X = x$.

Answer:

$$f_X(x) = \int_x^\infty 2\exp(-(x + y))dy = 2e^{-2x}, \qquad 0 < x < \infty$$

Then, the conditional p.d.f. is given as:

$$f_{Y|X}(y|x) = \frac{2\exp(-(x + y))}{2e^{-2x}} = e^{(x-y)}, \qquad 0 < x < y < \infty$$

with conditional mean

$$\mathbb{E}(Y|X = x) = \int_x^\infty ye^{x-y}dy = x + 1, \qquad x > 0$$