Stats 3D03

Solutions for Mid-term Test 1

Total: 15

1. (a) Pr(all cards are of the same suit)

$$= \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = \frac{4\binom{13}{5}}{\binom{52}{5}}.$$

(1 mark)

(b) Pr(hand includes four of a kind)

$$= \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{13 \times 48}{\binom{52}{5}}.$$

(1 mark)

(c) Pr(hand includes 3 Jacks, 1 Queen, 1 King)

$$= \frac{\binom{4}{3}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{64}{\binom{52}{5}}.$$

(1 mark)

2. (a)

$$\mathcal{C} = \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{array} \right\}$$

(1 mark)

(b) $\Pr(\text{total is an odd number})$

$$= \Pr\{(1,2), (1,4), (1,7), (2,1), (2,3), (2,5), \ldots, (6,5)\}$$

$$= \frac{18}{36} = \frac{1}{2}.$$

(1.5 marks)

(c) $\Pr(5 \mid \text{odd}) = \dfrac{\Pr(\text{Odd and } 5)}{\Pr(\text{Odd})}$

$$= \frac{\Pr\{(1,4), (2,3), (3,2), (4,1)\}}{18/36}$$

$$= \frac{4/36}{18/36} = \frac{4}{18} = \frac{2}{9}.$$

(1.5 marks)

3. $Pr(A) = 0.25$, $Pr(B) = 0.15$, $Pr(C) = 0.40$, $Pr(D) = 0.20$

$Pr(NC|A) = 0.02$, $Pr(NC|B) = 0.03$, $Pr(NC|C) = 0.01$, $Pr(NC|D) = 0.02$
$Pr(C|A) = 0.98$, $Pr(C|B) = 0.97$, $Pr(C|C) = 0.99$, $Pr(C|D) = 0.98$

(a) $Pr(C) = (0.25 \times 0.98) + (0.15 \times 0.97) + (0.40 \times 0.99) + (0.20 \times 0.98)$

$= 0.2450 + 0.1455 + 0.3960 + 0.1960$

$= 0.9825$ (1 mark)

(b) Pr(came from C | conforming)

$$= \frac{Pr(\text{from C and conforming})}{Pr(\text{conforming})}$$

$$= \frac{0.3960}{0.9825} = 0.4031$$

($1\frac{1}{2}$ marks)

(c) Pr(Non-conforming) = 1 − Pr(conforming)

$= 1 - 0.9825$

$= 0.0175$ ($\frac{1}{2}$ mark)

Pr(came from C | non-conforming)

$$= \frac{Pr(\text{from C and non-conforming})}{Pr(\text{non-conforming})}$$

$$= \frac{0.40 \times 0.01}{0.0175}$$

$= 0.2286$ (1 mark)

4. $Pr(H) = \frac{2}{3}$; $Pr(T) = \frac{1}{3}$.

(a) Z = No. of Heads in 3 tosses.

$$Pr(Z=0) = \binom{3}{0}\left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$Pr(Z=1) = \binom{3}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{6}{27}$$

$$Pr(Z=2) = \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{12}{27}$$

$$Pr(Z=3) = \binom{3}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = \frac{8}{27}$$

($1\frac{1}{2}$ marks)

(b) $Pr(Z \geq 2)$ = Pr(at least 2 Heads)

$$= \frac{12}{27} + \frac{8}{27} = \frac{20}{27}.$$

(1 mark)

(c) $$E(Z) = \left(0 \times \frac{1}{27}\right) + \left(1 \times \frac{6}{27}\right) + \left(2 \times \frac{12}{27}\right) + \left(3 \times \frac{8}{27}\right)$$

$$= \frac{54}{27} = 2$$

$$E(Z^2) = \left(0^2 \times \frac{1}{27}\right) + \left(1^2 \times \frac{6}{27}\right) + \left(2^2 \times \frac{12}{27}\right) + \left(3^2 \times \frac{8}{27}\right)$$

$$= \frac{6+48+72}{27} = \frac{126}{27} = \frac{14}{3}$$

$$Var(Z) = E(Z^2) - (EZ)^2 = \frac{14}{3} - 2^2 = \frac{2}{3}.$$

($1\frac{1}{2}$ marks)

5. $f(x) = \frac{c}{\theta} e^{-x/\theta}, \quad x>0, \theta>0.$

(a) $$\int_0^\infty f(x)\,dx = c\int_0^\infty \frac{1}{\theta} e^{-x/\theta}\,dx = -c\int_0^\infty \frac{d}{dx}\left(e^{-x/\theta}\right)$$

$$= -c\, e^{-x/\theta}\Big|_0^\infty = c = 1$$

$$\Rightarrow c = 1$$ (1 mark)

(b) $$F(x) = \int_0^x f(t)\,dt = \int_0^x \frac{1}{\theta} e^{-t/\theta}\,dt = -\int_0^x \frac{d}{dt}\left(e^{-t/\theta}\right)$$

$$= -e^{-t/\theta}\Big|_0^x = 1 - e^{-x/\theta}, \text{ for } x>0.$$

(1 mark)

(c) $$E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty x\cdot\frac{1}{\theta} e^{-x/\theta}\,dx = \int_0^\infty -x \frac{d}{dx}\left(e^{-x/\theta}\right)$$

$$= -x e^{-x/\theta}\Big|_0^\infty + \int_0^\infty e^{-x/\theta}\,dx$$

$$= 0 + \theta\int_0^\infty \frac{1}{\theta} e^{-x/\theta}$$

$$= \theta.$$ (1 mark)

(d) $$\Pr(X \leq 1) = 1 - e^{-1/\theta} = 0.01$$

$$\Rightarrow e^{-1/\theta} = 1 - 0.01 = 0.99$$

$$\Rightarrow -1/\theta = \ln(0.99)$$

$$\Rightarrow \theta = -\frac{1}{\ln(0.99)}$$

So, for at most 1% of the units to be returned within 1 year, θ should be at least $-\frac{1}{\ln(0.99)} = 99.499$.

(1 mark)