Stats 3D03 — Mid-Term Test 2

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Instructor: Prof. N. Balakrishnan **Total Marks: 15** **Duration: 75 mins.**

(1) Suppose the joint density function of X and Y is

$$f(x, y) = C(y - x), \quad 0 < x < y < 1.$$

Then:

(a) Find the normalizing constant C that would make the above $f(x, y)$ to be a valid density function;

(b) Derive the marginal density functions of X and Y;

(c) Derive the conditional density functions of $X \mid (Y = y)$ and of $Y \mid (X = x)$;

(d) Prove that $\frac{X}{Y}$ and Y are independent;

(e) Use Part (d) to determine $Cov(X, Y)$.

(5 marks)

(2) Suppose $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$ are independent random variables. Then:

(a) Find the moment generating function of X;

(b) Deduce from Part (a) the mean and variance of X;

(c) Using Part (a), identify the distribution of $X + Y$.

(3 marks)

(3) Suppose $X \sim \chi^2_{\nu_1}$ and $Y \sim \chi^2_{\nu_2}$ are independent random variables. Then:

(a) Derive an expression for $E(X^r)$ (the r^{th} moment of X) and discuss when it will exist;

(b) If $U = \frac{X}{X+Y}$ and $V = X + Y$, derive the joint density function of U and V;

(c) Identify the distributions of U and V;

(d) Are U and V independent, and explain why or why not?

(4 marks)

(4) Suppose $X \sim \chi^2_{\nu_1}$ and $Y \sim \chi^2_{\nu_2}$ are independent random variables, and $W = \frac{X/\nu_1}{Y/\nu_2}$. Then, W is said to have a F-distribution with degrees of freedom (ν_1, ν_2).

(a) Derive the probability density function of W;

(b) Find $E(W)$ (the mean) and comment.

(4 marks)

(5) Suppose $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ are random samples from $N(\mu_X, \sigma^2)$ and $N(\mu_Y, \sigma^2)$ distributions, respectively. Our interest is in constructing $100(1-\alpha)\%$ confidence interval for the mean difference $\mu_X - \mu_Y$. Then:

(a) Explain the construction of the pivot (outline the steps of the process);

(b) What is the distribution of the pivot?

(c) Use Parts (a) and (b) to derive an exact $100(1-\alpha)\%$ confidence interval for $\mu_X - \mu_Y$;

(d) In the construction of the pivot, if you use a pooled estimate of σ^2 (by pooling the estimates of σ^2 from the two samples), provide a justification as to why that pooling is the best!

(4 marks)

Good luck!

Stats 3D03

Solutions for Mid-term Test 2

1.(a) $\displaystyle\int_{x=0}^{1}\int_{y=x}^{1} f(x,y) = C\int_{x=0}^{1} \frac{(y-x)^2}{2}\Big|_x^1 dx = \frac{C}{2}\int_{x=0}^{1}(1-x)^2\,dx$

$\displaystyle = -\frac{C}{6}(1-x)^3\Big|_0^1 = \frac{C}{6} = 1$

$\Rightarrow C = 6.$ (1 mark)

(b) $\displaystyle f_X(x) = 6\int_x^1 (y-x)\,dy = 6\frac{(y-x)^2}{2}\Big|_x^1 = 3(1-x)^2,\ 0<x<1$

$\displaystyle f_Y(y) = 6\int_0^y (y-x)\,dx = -6\frac{(y-x)^2}{2}\Big|_0^y = 3y^2,\ 0<y<1.$ (1 mark)

(c) $\displaystyle f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{6(y-x)}{3y^2} = \frac{2(y-x)}{y^2},\ 0<x<y.$

$\displaystyle f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{6(y-x)}{3(1-x)^2} = \frac{2(y-x)}{(1-x)^2},\ x<y<1$ (1 mark)

(d) Let $U = \frac{X}{Y}$ and $V = Y$

$\Rightarrow X = UY = UV$ and $Y = V$

$|J| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$

$\therefore f_{U,V}(u,v) = 6(v-uv)v = 6v^2(1-u),\ 0<u<1,\ 0<v<1$

$\Rightarrow U$ and V are independent by Factorization Thm. (1 mark)

(e) $\displaystyle E(X) = \int_0^1 3x(1-x)^2\,dx = 3B(2,3) = 3\cdot\frac{1!\,2!}{4!} = \frac{1}{4}$

$\displaystyle E(Y) = \int_0^1 3y^3\,dy = \frac{3}{4}.$ (1 mark)

$\displaystyle E(U) = 2\int_0^1 u(1-u)\,du = 2B(2,2) = 2\frac{1!\,1!}{3!} = \frac{1}{3}$

$\displaystyle E(V^2) = 3\int_0^1 v^4\,dv = \frac{3}{5}.\ \Rightarrow E(XY) = E(UV^2) = \frac{1}{3}\times\frac{3}{5} = \frac{1}{5}$

$\displaystyle \Rightarrow Cov(X,Y) = \frac{1}{5} - \frac{1}{4}\times\frac{3}{4} = \frac{1}{5} - \frac{3}{16} = \frac{16-15}{80} = \frac{1}{80}$

2. $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$.

(a) $f_X(x) = e^{-\lambda_1} \lambda_1^x / x!$, $x = 0, 1, \ldots$

$$M_X(t) = E(e^{tX}) = e^{-\lambda_1} \sum_{x=0}^{\infty} (\lambda_1 e^t)^x / x! = e^{-\lambda_1 + \lambda_1 e^t}$$

(1 mark)

(b) $\frac{d}{dt} M_X(t) = e^{-\lambda_1} \cdot e^{\lambda_1 e^t} \lambda_1 e^t \Rightarrow \text{Mean} = \frac{d}{dt} M_X(t)\Big|_{t=0} = \lambda_1$

$$\frac{d^2}{dt^2} M_X(t) = e^{-\lambda_1} e^{\lambda_1 e^t} (\lambda_1 e^t)^2 + e^{-\lambda_1} e^{\lambda_1 e^t} \lambda_1 e^t$$

$$\Rightarrow E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \lambda_1^2 + \lambda_1$$

$$\Rightarrow \text{Var}(X) = E(X^2) - (EX)^2 = \lambda_1^2 + \lambda_1 - \lambda_1^2 = \lambda_1.$$

(1 mark)

(c) Let $Z = X + Y$.

MGF of Z is

$$M_Z(t) = E(e^{tZ}) = E(e^{tX} \cdot e^{tY}) = E(e^{tX}) E(e^{tY})$$

$$= e^{-\lambda_1 + \lambda_1 e^t} \cdot e^{-\lambda_2 + \lambda_2 e^t}$$

$$= e^{-(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2) e^t}$$

$$\Rightarrow Z = X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2).$$

(1 mark)

3. $X \sim \chi^2_{\nu_1}$, $Y \sim \chi^2_{\nu_2}$, and are independent.

(a) $$E(X^r) = \frac{1}{2^{\nu_1/2}\Gamma(\nu_1/2)} \int_0^\infty e^{-\frac{x}{2}} x^{\frac{\nu_1}{2}+r-1}\,dx$$

$$= \frac{1}{2^{\nu_1/2}\Gamma\left(\frac{\nu_1}{2}\right)} 2^{\frac{\nu_1}{2}+r}\,\Gamma\left(\frac{\nu_1}{2}+r\right)$$

$$= 2^r\,\Gamma\left(\frac{\nu_1}{2}+r\right)\Big/\Gamma\left(\frac{\nu_1}{2}\right).$$

($1\frac{1}{2}$ marks)

It exists whenever $\frac{\nu_1}{2}+r>0$.

(b) $$f_{X,Y}(x,y) = \frac{1}{2^{\nu_1/2}2^{\nu_2/2}\Gamma(\nu_1/2)\Gamma(\nu_2/2)} e^{-\frac{x}{2}} e^{-\frac{y}{2}} x^{\frac{\nu_1}{2}-1} y^{\frac{\nu_2}{2}-1},$$

$0<x, y<\infty$.

$$U = \frac{X}{X+Y},\quad V = X+Y$$

$$\Rightarrow X = UV,\quad Y = V-X = V-UV = V(1-U)$$

$$|J| = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v-uv+uv = v.$$

$$f_{U,V}(u,v) = \frac{1}{2^{\frac{\nu_1+\nu_2}{2}}\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} e^{-\frac{v}{2}} u^{\frac{\nu_1}{2}-1} v^{\frac{\nu_1}{2}-1} v^{\frac{\nu_2}{2}-1} (1-u)^{\frac{\nu_2}{2}-1} v$$

$$= \frac{1}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} u^{\frac{\nu_1}{2}-1}(1-u)^{\frac{\nu_2}{2}-1} \cdot \frac{1}{2^{\frac{\nu_1+\nu_2}{2}}\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)} e^{-\frac{v}{2}} v^{\frac{\nu_1+\nu_2}{2}-1},$$

$0<u<1$, $0<v<1$

($1\frac{1}{2}$ marks)

(c) $U \sim \text{Beta}\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)$ and $V \sim \chi^2_{\nu_1+\nu_2}$. ($\frac{1}{2}$ mark)

(d) By factorization theory, U and V are independent random variables. ($\frac{1}{2}$ mark)

4. $X \sim \chi^2_{\nu_1}$, $Y \sim \chi^2_{\nu_2}$ and are independent. [4]

(a) $$f_{X,Y}(x,y) = \frac{1}{2^{\nu_1/2} 2^{\nu_2/2} \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} e^{-\frac{x}{2}} x^{\frac{\nu_1}{2}-1} e^{-\frac{y}{2}} y^{\frac{\nu_2}{2}-1}, \quad 0<x,y<\infty.$$

Let $W = \frac{\nu_2}{\nu_1} \cdot \frac{X}{Y}$ and $V = Y$.

$\Rightarrow X = \frac{\nu_1}{\nu_2} \cdot WV$ and $Y = V$.

$$|J| = \begin{vmatrix} \frac{\nu_1}{\nu_2} v & \frac{\nu_1}{\nu_2} w \\ 0 & 1 \end{vmatrix} = \frac{\nu_1}{\nu_2} v.$$

$$\Rightarrow f_{W,V}(w,v) = \frac{1}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} e^{-\frac{1}{2}\frac{\nu_1}{\nu_2} wv - \frac{1}{2}v} \left(\frac{\nu_1}{\nu_2} wv\right)^{\frac{\nu_1}{2}-1} v^{\frac{\nu_2}{2}-1} \cdot \frac{\nu_1}{\nu_2} v$$

$$= \frac{1}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} e^{-\frac{1}{2}(1+\frac{\nu_1}{\nu_2} w)v} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} w^{\frac{\nu_1}{2}-1} v^{\frac{\nu_1+\nu_2}{2}-1}, \quad 0<w,v<\infty.$$

$$\Rightarrow f_W(w) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} w^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \int_0^\infty e^{-\frac{1}{2}(1+\frac{\nu_1}{\nu_2} w)v} v^{\frac{\nu_1+\nu_2}{2}-1} dv$$

$$= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} w^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \cdot \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\left(\frac{1}{2}(1+\frac{\nu_1}{\nu_2} w)\right)^{\frac{\nu_1+\nu_2}{2}}}$$

$$= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} w^{\frac{\nu_1}{2}-1}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)\left(1+\frac{\nu_1}{\nu_2} w\right)^{\frac{\nu_1+\nu_2}{2}}}, \quad 0<w<\infty.$$

(3 marks)

(b) $E(W) = \frac{\nu_2}{\nu_1} E\left(\frac{X}{Y}\right) = \frac{\nu_2}{\nu_1} E(X) E\left(\frac{1}{Y}\right)$ due to independence

$$= \frac{\nu_2}{\nu_1} \frac{2\Gamma(\frac{\nu_1}{2}+1)}{\Gamma(\frac{\nu_1}{2})} \frac{2^{-1}\Gamma(\frac{\nu_2}{2}-1)}{\Gamma(\frac{\nu_2}{2})} = \frac{\nu_2}{\nu_1} \cdot 2 \frac{\nu_1}{2} \cdot \frac{1}{2} \frac{1}{(\frac{\nu_2}{2}-1)}$$

$= \frac{\nu_2}{\nu_2 - 2}$ which exists when $\nu_2 > 2$.

Note it is not 1, but tends to 1 when $\nu_2 \to \infty$. (1 mark)

5. $X_1 \dots X_m \rightarrow N(\mu_x, \sigma^2)$

(a) $Y_1, \dots, Y_n \rightarrow N(\mu_y, \sigma^2)$.

So, $\bar{X} \rightarrow N(\mu_x, \frac{\sigma^2}{m})$, $\frac{(m-1)S_x^2}{\sigma^2} \rightsquigarrow \chi^2_{m-1}$

$\bar{Y} \rightarrow N(\mu_y, \frac{\sigma^2}{n})$, $\frac{(n-1)S_y^2}{\sigma^2} \rightsquigarrow \chi^2_{n-1}$

All are independent.

$\Rightarrow \bar{X}-\bar{Y} \rightarrow N(\mu_x-\mu_y, \sigma^2(\frac{1}{m}+\frac{1}{n}))$

$\Rightarrow \frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{\sigma^2(\frac{1}{m}+\frac{1}{n})}} \rightarrow N(0,1)$.

(b) Pooled variance $S_p^2 = \{(m-1)S_x^2+(n-1)S_y^2\}/(m+n-2)$

So, $(m+n-2)\frac{S_p^2}{\sigma^2} \rightarrow \chi^2_{m+n-2}$.

(1 mark)

$\Rightarrow \frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{\sigma^2(\frac{1}{m}+\frac{1}{n})}} \Big/ \sqrt{\frac{(m+n-2)S_p^2}{\sigma^2}/(m+n-2)}$

$\rightarrow t_{m+n-2}$

$\Rightarrow \frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{S_p^2(\frac{1}{m}+\frac{1}{n})}} \rightarrow t_{m+n-2}$.

(1 mark)

(c) An exact $100(1-\alpha)\%$ CI for $\mu_x-\mu_y$ is

$\left(\bar{X}-\bar{Y}-t_{m+n-2,\frac{\alpha}{2}}\sqrt{S_p^2(\frac{1}{m}+\frac{1}{n})},\ \bar{X}-\bar{Y}+t_{m+n-2,\frac{\alpha}{2}}\sqrt{S_p^2(\frac{1}{m}+\frac{1}{n})}\right)$

(1 mark)

(d) Let pooled variance be $wS_x^2+(1-w)S_y^2$.

Then, $E(wS_x^2+(1-w)S_y^2) = w\sigma^2+(1-w)\sigma^2 = \sigma^2$.

Next, $Var(wS_x^2+(1-w)S_y^2) = w^2\frac{2\sigma^4}{m-1}+(1-w)^2\frac{2\sigma^4}{n-1}$.

Differentiating w.r.t. w, we get $2\sigma^4\left(\frac{2w}{m-1}-\frac{2(1-w)}{n-1}\right)=0$

$\Rightarrow w = \frac{m-1}{m+n-2}$. Hence, S_p^2.

(1 mark)