1. Exercise 4.1.3 on Page 213

Suppose the number of customer X that enter a store between the hours 9:00 a.m. and 10:00 a.m. follows a Poisson distribution with parameter θ. Suppose a random sample of the number of customers that enter the store between 9:00 a.m. and 10:00 a.m. for 10 days results in the values

$$9 \quad 7 \quad 9 \quad 15 \quad 10 \quad 13 \quad 11 \quad 7 \quad 2 \quad 12$$

(a) Determine the maximum likelihood estimate of θ. Show that it is an unbiased estimator.
Answer (a):

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{10} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-10\lambda}\lambda^{\sum_{i=1}^{10} x_i}}{\prod_{i=1}^{10} x_i!} \\ l(\theta) &= -10\lambda + \sum_{i=1}^{10} x_i \log(\lambda) - \log(\prod_{i=1}^{10} x_i!) = 0 \\ \lambda &= \frac{\sum_{i=1}^{10} x_i}{10} \end{aligned} \tag{1}$$

(b) Based on these data, obtain the realization of your estimator in part(a). Explain the meaning of this estimate in terms of the number of customers.

$$\begin{aligned} \lambda &= \frac{\sum_{i=1}^{10} x_i}{10} = 9.5 \\ \text{As} \qquad \mathbb{E}(X) &= \lambda = 9.5 \end{aligned} \tag{2}$$

Therefore, this means the average number of customers coming is 9.5.

2. Exercise 4.1.7 on Pages 213-214

The data set on Scottish schoolchildren discussed in Example 4.1.5 included the eye colours of the children also. The frequencies of their eye colours are::

Blue	Light	Medium	Dark
2978	6697	7511	5175

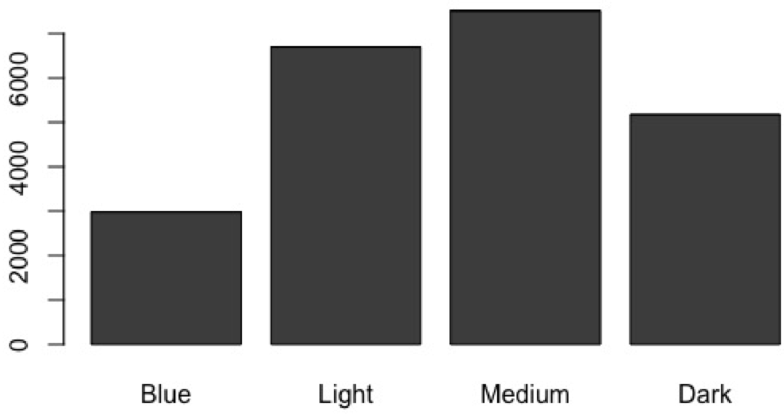
Use these frequencies to obtain a bar chart and an estimate of the associated p.m.f.
Answer:

```
> ##Data
> a11<-c(2978,6697,7511,5175)
> a21<-a11/sum(a11)
> amatrix<-rbind(a11,a21)
```

```
> colnames(amatrix)<-c('Blue','Light','Medium','Dark')
> ##Barchart
> barplot(amatrix)
```

Table 1: Table

Blue	Light	Medium	Dark
2978	6697	7511	5175
0.1331783	0.2994947	0.3358973	0.2314297



3. Exercise 4.2.2 on Page 219

Consider the data on the lifetimes of motors given in Exercise 4.1.1. Obtain a large sample of confidence intervals for the mean lifetime of motor.
Where we assume that the lifetime of a motor under these conditions, X, has a $\Gamma(1, \theta)$ distribution:

1	4	5	21	22	28	40	42	51	53
58	67	95	124	124	160	202	260	303	363

Answer:

```
> ##data
> aaa<-c(1,4,5,21,22,28,40,42,51,53,58,67,95,124,124,160,202,260,303,363)
```

```
> ##length of data
> n<-length(aaa)
> ##mean
> mu_a<-mean(aaa)
> ##standard deviation
> s<-sd(aaa)
> ##margin of error 95%
> error <- qnorm(0.975)*s/sqrt(n)
> ##confidence interval
> left <- mu_a-error
> right <- mu_a+error
> c(left,right)

[1]  54.95327 147.34673
```

4. Exercise 4.2.7 on Page 220

Let $\bar{X}$ be the mean of a random sample of size n from a distribution that is $N(\mu, 9)$. Find n such that

$$P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$$

approximately.

Answer:

$$\begin{aligned}
&P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90 \\
&P\Big(\frac{\bar{X} - 1 - \mu}{\sigma/\sqrt{n}} < 0 < \frac{\bar{X} + 1 - \mu}{\sigma/\sqrt{n}}\Big) = 0.90 \\
&P\Big(Z - \frac{1}{\sigma/\sqrt{n}} < 0 < Z + \frac{1}{\sigma/\sqrt{n}}\Big) = 0.90 \\
&P\Big(-\frac{1}{\sigma/\sqrt{n}} < Z < \frac{1}{\sigma/\sqrt{n}}\Big) = 0.90 \\
&\frac{1}{3/\sqrt{n}} = 1.645 \Rightarrow n = (1.645 \times 3)^2 = 24.35423
\end{aligned} \tag{3}$$

5. Exercise 4.2.10 on Page 221

Let $X_1, X_2, ..., X_9$ be a random sample of size 9 from a distribution that is $N(\mu, \sigma^2)$.

(a) If σ is known, find the length of a 95% confidence interval for μ if this interval is based on the random variable $\sqrt{9}(\bar{X} - \mu)/\sigma$

Answer(a):

$$\begin{aligned} &P(-1.96 < Z = \sqrt{9}(\bar{X} - \mu)/\sigma < 1.96) = 0.95 \\ \text{Expected length:} \quad &2 \times 1.96 \times \frac{\sigma}{3} = 1.306667\sigma \end{aligned} \tag{4}$$

(b) If σ is unknown, find the expected value of the length of a 95% confidence interval for μ if this interval is based on the random variable $\sqrt{9}(\bar{X} - \mu)/S$. (Hint: Write $\mathbb{E}(S) = (\sigma/\sqrt{n-1})\mathbb{E}[(\frac{(n-1)S^2}{\sigma^2})^{1/2})]$

Answer(b): First, remember that when σ is unknown with sample size =9, which is rather small, the $\sqrt{9}(\bar{X} - \mu)/S$ follows a t distribution with $n-1$ degree of freedom.

Meanwhile, recall the fact that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

where

$$p(y) = \frac{(1/2)^{(n-1)/2}}{\Gamma((n-1)/2)} y^{(n-1)/2-1} e^{-y/2}$$

Then, we have

$$\bar{X} \pm \frac{2.306S}{\sqrt{9}}$$

We want to calculate the expected length

$$\mathbb{E}(2 \times \frac{2.306S}{\sqrt{9}})$$

Thus, we need

$$\begin{aligned} \mathbb{E}(S) &= \frac{\sigma}{\sqrt{n-1}} \mathbb{E}[(\frac{(n-1)S^2}{\sigma^2})^{1/2}] \\ &= \frac{\sigma}{\sqrt{n-1}} \mathbb{E}[\sqrt{y}] \\ &= \frac{\sigma}{\sqrt{n-1}} \int_0^\infty \sqrt{y} \frac{(1/2)^{(n-1)/2}}{\Gamma((n-1)/2)} y^{(n-1)/2-1} e^{-y/2} dy \\ &= \frac{\sigma}{\sqrt{n-1}} \times \frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})} \frac{(1/2)^{(n-1)/2}}{(1/2)^{n/2}} \int_0^\infty \frac{(1/2)^{n/2}}{\Gamma(n/2)} y^{n/2-1} e^{-y/2} dy \\ &= \frac{\sigma}{\sqrt{n-1}} \times \frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})} \times \sqrt{2} \\ \text{when } (n=9) \quad &= 0.9693107\sigma \end{aligned} \tag{5}$$

Hence, the expected length:

$$\mathbb{E}(2 \times \frac{2.306S}{\sqrt{9}}) = 1.490154\sigma$$

6. Exercise 4.2.18 on Page 221

Let $X_1, X_2, ..., X_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters μ and σ^2 are known. A confidence interval for σ^2 can be found as follows. We know that $(n-1)S^2/\sigma^2$ is a random variable with a $\chi^2(n-1)$ distribution. Thus we can find constants a and b so that $P((n-1)S^2/\sigma^2 < b) = 0.975$ and $P(a < (n-1)S^2/\sigma^2 < b) = 0.95$.

(a) Show that this second probability statement can be written as:

$$P(\frac{(n-1)S^2}{b} < \sigma^2 < \frac{(n-1)S^2}{a}) = 0.95$$

Answer(a): This is obvious.

(b) If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for σ^2.

Answer(b): From (a), we have$a = 2.179731, b = 17.53455$

$$P(\frac{(9-1)\times 7.93}{17.53455} < \sigma^2 < \frac{(9-1)\times 7.93}{2.179731}) = 0.95$$

$$P(3.618 < \sigma^2 < 29.10451) = 0.95$$

(c) If μ is known, how would you modify the preceding procedure for finding a confidence interval for σ^2?

Answer(c):

We can use

$$\frac{\sum(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$$

Histgram and density plot

```
> library(gdata)
> df_1 = read.xls ("Stats3d_marking.xls", sheet = 1, header = TRUE)
> dat11<-df_1$Test.1
> length(dat11)

[1] 139

> length(na.omit(dat11))

[1] 126

> mean(na.omit(dat11))

[1] 13.812

> sd(na.omit(dat11))
```

```
[1] 3.26476
```

```
> par(mfrow=c(1,2))
> hist(dat11)
> plot(density(na.omit(dat11)))
```