Correction to Example done today in class

Sorry I made an error starting the example in class today and that snowballed so here is the corrected example.

Setup: We want to generate $X \sim \text{Gamma}(\alpha, \beta)$ $(\alpha \geqslant 1)$ using accept-reject sampling from an exponential candidate density.

Let λ be the rate parameter of the candidate density. Then the ratio of densities is

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{1}{\Gamma(\alpha)\beta^\alpha \lambda} x^{\alpha-1} \exp\left\{-\left(\frac{1}{\beta} - \lambda\right) x\right\} \\ &\propto x^{\alpha-1} \exp\left\{-\left(\frac{1}{\beta} - \lambda\right) x\right\} \end{aligned}$$

At this point let us note that the ratio is not always bounded. If $\lambda > 1/\beta$ then the exponent will be e^{cx} for some positive c which will grow without bound. Also if $\lambda = 1/\beta$ then the exponent is 1 but $x^{\alpha-1}$ grows without bound so the ratio is still unbounded. Therefore we require $\lambda < 1/\beta$ for the algorithm to work.
Assuming that $\lambda < 1/\beta$, to maximize the ratio of densities we first take logs and find the value of x (call it x_0) that maximizes the log ratio.

$$\begin{aligned} \log \frac{f(x)}{g(x)} &= K + (\alpha - 1)\log x - \left(\frac{1}{\beta} - \lambda\right) x \\ \Rightarrow \quad \frac{d}{dx} \log \frac{f(x)}{g(x)} &= \frac{\alpha - 1}{x} - \frac{1}{\beta} - \lambda \\ \Rightarrow \quad 0 &= \frac{\alpha - 1}{x_0} - \left(\frac{1}{\beta} - \lambda\right) \\ \Rightarrow \quad x_0 &= \frac{\alpha - 1}{\frac{1}{\beta} - \lambda} = \frac{(\alpha - 1)\beta}{1 - \lambda\beta} \end{aligned}$$

Now we can plug this into the ratio of densities to find the maximum as a function of λ

$$\begin{aligned} M(\lambda) = \frac{f(x_0)}{g(x_0)} &= \frac{1}{\Gamma(\alpha)\beta^\alpha \lambda} x_0^{\alpha-1} \exp\left\{-\left(\frac{1}{\beta} - \lambda\right) x_0\right\} \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha \lambda} \left(\frac{(\alpha - 1)\beta}{1 - \lambda\beta}\right)^{\alpha-1} \mathrm{e}^{1-\alpha} \\ &= \frac{1}{\Gamma(\alpha)\lambda\beta} \left(\frac{\alpha - 1}{1 - \lambda\beta}\right)^{\alpha-1} \mathrm{e}^{1-\alpha} \\ &\propto \frac{1}{\lambda} \times \frac{1}{(1 - \lambda\beta)^{\alpha-1}} = h(\lambda) \end{aligned}$$

Next we will find the value of λ (lets call it $\hat{\lambda}$) which minimizes the function $h(\lambda)$ and hence minimizes $M(\lambda)$.

$$\begin{aligned}
\log h(\lambda) &= -\log\lambda - (\alpha-1)\log(1-\lambda\beta) \\
\Rightarrow \quad \frac{d}{d\lambda}\log h(\lambda) &= -\frac{1}{\lambda} + \frac{(\alpha-1)\beta}{1-\lambda\beta} \\
\Rightarrow \quad 0 &= -\frac{1}{\hat{\lambda}} + \frac{(\alpha-1)\beta}{1-\hat{\lambda}\beta} \\
\Rightarrow \quad \frac{1}{\hat{\lambda}} &= \frac{(\alpha-1)\beta}{1-\hat{\lambda}\beta} \\
\Rightarrow \quad 1-\hat{\lambda}\beta &= \hat{\lambda}(\alpha-1)\beta \\
\Rightarrow \quad 1-\hat{\lambda}\beta &= \hat{\lambda}\alpha\beta - \hat{\lambda}\beta \\
\Rightarrow \quad \hat{\lambda} &= \frac{1}{\alpha\beta}
\end{aligned}$$

We note that the mean of the target density is $\alpha\beta$ and so this optimal choice of λ says we should use the exponential with the same mean as the target in this example.
The actual bound for this choice of λ is

$$\begin{aligned}
M(\hat{\lambda}) &= \frac{1}{\Gamma(\alpha)\hat{\lambda}\beta}\left(\frac{\alpha-1}{1-\hat{\lambda}\beta}\right)^{\alpha-1} \mathrm{e}^{1-\alpha} \\
&= \frac{1}{\Gamma(\alpha)\frac{1}{\alpha}}\left(\frac{\alpha-1}{1-\frac{1}{\alpha}}\right)^{\alpha-1} \mathrm{e}^{1-\alpha} \\
&= \frac{\alpha}{\Gamma(\alpha)}\left(\frac{\alpha(\alpha-1)}{\alpha-1}\right)^{\alpha-1} \mathrm{e}^{1-\alpha} \\
&= \frac{\alpha^{\alpha}}{\Gamma(\alpha)}\mathrm{e}^{1-\alpha}
\end{aligned}$$

Hence we can code this using the code in the file `Jan31.R`.