STATISTICS 4CI3/6CI3

Winter 2019 Dr. Angelo Canty

TERM TEST March 7, 2019.

DAY CLASS
DURATION OF EXAMINATION: 50 Minutes

THIS EXAMINATION PAPER INCLUDES 2 PAGES AND 5 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Instuctions:

1. Use of the McMaster Standard Calculator (Casio FX-991 or fx-991MS) **only** is allowed.
2. All 5 questions carry 10 marks.
3. Answer all questions in the answer booklet clearly labelling each question number.
4. Ensure that you write your full name and student number on your answer booklet and any rough work paper that you use.

Q. 1 Suppose that we only have a source of Uniform$(0, 1)$ random variates. Describe an algorithm to generate independent random variates with probability mass function

$$f(x) = 0.25|x - 2.2| \qquad x = 1, 2, 3, 4$$

Use your algorithm to generate the 10 random variabtes corresponding to the following 10 uniform(0,1) random variates.

0.5197 0.1790 0.9994 0.2873 0.7294 0.5791 0.0361 0.3281 0.2026 0.8213

Q. 2 Consider the following `R` function.

```
rlaplace <- function(n) {
    U1 <- runif(n)
    U2 <- runif(n)
    X <- -log(U1)
    Y <- ifelse(U2<0.5, X, -X)
    Y
}
```

Prove that this function results in observations from the standard Laplace distribution with probability density function

$$g(y) = \frac{1}{2}e^{-|y|} \qquad y \in \mathbb{R}$$

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Q. 3 Suppose that we wish to sample from a standard normal distribution using the accept-reject algorithm from the standard Laplace distribution with density as generated in Question 2.

Show that $x^2 - 2|x| \geqslant -1$ and hence prove that, if $f(x)$ is the standard normal density function,

$$f(x) \;=\; \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \qquad x \in \mathbb{R}$$

then

$$\frac{f(x)}{g(x)} \;\leqslant\; \sqrt{\frac{2\mathrm{e}}{\pi}}$$

Give an algorithm to generate a standard normal observation using the accept-reject algorithm with a Laplace candidate distribution. You may assume existence of functions to generate uniform and Laplace random variables.

Q. 4 Consider Monte Carlo estimation of the integral

$$I \;=\; \int_0^\infty x^2 \mathrm{e}^{-x^2}\, dx$$

Use a transformation to show that I can be written as

$$I \;=\; \int_0^1 \frac{\sqrt{-\log(u)}}{2}\, du$$

Give the regular Monte Carlo estimator of I **using only uniform**$(0,1)$ **random numbers** and derive its standard error.

Q. 5 Consider a sample of size n from a gamma$(\alpha, 1)$ distribution with density function

$$f(x;\alpha) \;=\; \frac{1}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-x}$$

We wish to test the hypotheses $H_0 : \alpha = 1 \quad \text{V} \quad H_1 : \alpha > 1$ and decide to reject H_0 if the sample median is bigger than $1 + 10/n$.

An important characteristic of a test is the power curve which traces the probability of rejecting the null hypothesis for different values of the parameter. Carefully describe a simulation study to estimate the power curve of this test for $n = 20$ assuming that there is a function which will generate gamma random variables.

The End