STAT 743 Foundations of Statistics (Term 1)

Angelo J. Canty

Office : Hamilton Hall 209

Phone : (905) 525-9140 extn 27079

E-mail : cantya@mcmaster.ca

Web Page :
http://www.math.mcmaster.ca/canty/teaching/stat743

Course Information

Office Hours I will generally be in my office for 30–45 minutes after each class. Appointments can also be made by e-mail. For short questions, you can drop by my office and if I am there and free I will answer your question.

Assignments 4 assignments will be worth 60% of your mark (15% each) for Term 1.

Final Exam 3 hour written exam in December will be worth the other 40% of your mark for Term 1.

Course Grade Based on an equally weighted average of marks in both terms.

Topics to be covered

Week 1 Probability

Week 2 Random Variables

Weeks 3–4 Common Distributions

Weeks 5–6 Bivariate and Multivariate Distributions

Weeks 7–8 Random Samples

Week 9 Convergence

Week 10 Generating Random Samples

Week 11 Sufficient Statistics

Week 12 Introduction to Point Estimation

Probability

Definition 1.1
A random experiment is a process resulting in an outcome belonging to a well-defined set of possible outcomes. The outcome of any one run of the process, however, cannot be known in advance.

Definition 1.2
The set of all possible outcomes of a random experiment is called the sample space.

Definition 1.3
An event is any subset of the sample space. An event is said to occur if the outcome of the random experiment is an element of the event.

Set Operators

Suppose that A and B are two events in a sample space S.

Union $A \bigcup B = \{x \mid x \in A \text{ OR } x \in B\}$.

Intersection $[A \bigcap B = \{x \mid x \in A \text{ AND } x \in B\}$.

Complementation $A^c = \{x \mid x \in S \text{ AND } x \notin A\}$.

Definition 1.4

A sequence of events $A_1, A_2, \ldots$ are said to be mutually exclusive if

$$A_i \bigcap A_j \;=\; \emptyset. \qquad \textit{for every } i \neq j$$

Set Theory Results

Theorem 1.1

Suppose that A, B *and* C *are events in a sample space* S.

Commutative: $A \bigcup B = B \bigcup A \quad A \bigcap B = B \bigcap A$;

Associative:

$$A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C \quad A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C;$$

Distributive

$$A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$$

$$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C);$$

De Morgan's Laws

$$(A \bigcup B)^c = A^c \bigcap B^c \qquad (A \bigcap B)^c = A^c \bigcup B^c.$$

Sigma Algebra

Definition 1.5

A Sigma Algebra $\mathcal{B}$ *is a collection of events in a sample space* S *satisfying*

1. $S \in \mathcal{B}$.
2. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$.
3. $A_1, A_2, \ldots \in \mathcal{B} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$.

If S is a finite or countable set then we will generally use the power set of S as the sigma algebra

$$\mathcal{B} = \{A \mid A \subseteq S\}.$$

If S is an uncountable set then the power set is too large to be useful so instead we will use the smallest sigma algebra which contains all open subsets of S.

The Axioms of Probability

Definition 1.6

Given a sample space S and associated sigma algebra $\mathcal{B}$, a probability function is a function P defined on the elements of $\mathcal{B}$ which satisfies

1. $P(A) \geqslant 0$ *for all* $A \in \mathcal{B}$.

2. $P(S) = 1$.

3. *If* $A_1, A_2, \ldots \in \mathcal{B}$ *are mutually exclusive then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Other Rules of Probability

Theorem 1.2

Given a sample space S *and associated sigma algebra* $\mathcal{B}$, *let* A *and* B *be arbitrary elements of* $\mathcal{B}$ *then*

1. $P(\emptyset) = 0$;
2. $P(A) \leqslant 1$;
3. $P(A^c) = 1 - P(A)$;
4. $P(A \bigcap B^c) = P(A) - P(A \bigcap B)$;
5. $\mathsf{P}(A \bigcup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \bigcap B)$;
6. $A \subset B \;\Rightarrow\; P(A) \leqslant P(B)$.

Partitions

Definition 1.7

A partition of a sample space S is a collection of events $A_1, A_2, \ldots$ satisfying

1. $A_i \bigcap A_j = \emptyset$ *for all* $i \neq j$,
2. $\displaystyle\bigcup_{i=1}^{\infty} A_i = S$.

Theorem 1.3

Suppose that S is a sample space with associated sigma algebra $\mathcal{B}$ and that P is a probability function on $\mathcal{B}$. Then for any partition $C_1, C_2, \ldots$ of S and any $A \in \mathcal{B}$,

$$P(A) = \sum_{i=1}^{\infty} P(A \bigcap C_i)$$

Boole's Inequality

Theorem 1.4

Suppose that S is a sample space with associated sigma algebra $\mathcal{B}$ and that P is a probability function on $\mathcal{B}$. For any events $A_1, A_2, \ldots \in \mathcal{B}$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} P(A_i).$$

Counting

* The number of ways that n distinct objects can be re-arranged is $n! = n(n-1)(n-2)\cdots 2\cdot 1$

* If the n objects are not all distinct but a collection of m distinct objects repeated $n_1, \ldots, n_m$ times ($n = \sum n_i$) then the number of distinct arrangements is

$$\frac{n!}{n_1!n_2!,\cdots n_m!}$$

* The number of ways of selecting an ordered set of r objects from n distinct objects without replacement is $n!/r!$.

* If the order of the r selected objects is irrelevant then the number of distinct sets is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Conditional Probability and Independence

Definition 1.8

Suppose that A and B are two events such that $P(B) > 0$ then the conditional probability that A occurs given that the event B occurs is defined as

$$P(A \mid B) = \frac{P(A \bigcap B)}{P(B)}$$

Theorem 1.5

For any event B with $P(B) > 0$, the conditional probability function $P(\cdot \mid B)$ satisfies the Axioms of Probability.

Bayes' Rule

Theorem 1.6

Suppose that $A_1, A_2, \ldots$ *is a partition of a sample space* S *and let* $B \in S$. *Then for each* $i = 1, 2, \ldots,$

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B \mid A_j)P(A_j)}$$

Independence

Definition 1.9

Two events A and B are statistically independent if, and only if,

$$P(A \bigcap B) = P(A)P(B)$$

Theorem 1.7

If A and B are independent events then so are the following pairs of events

1. A *and* B^c,
2. A^c *and* B,
3. A^c *and* B^c,

Independence

Definition 1.10

A collection of events $A_1, \ldots, A_n$ are mutually independent if, and only if, for every $\{i_1, \ldots, i_2\} \subseteq \{1, 2, \ldots, n\}$ the subcollection $A_{i_1}, \ldots, A_{i_k}$ satisfies

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j}).$$

If for every pair of events A_i, A_j with $i \neq j$ we have

$$P(A_i \bigcap A_j) = P(A_i)P(A_j)$$

then the collection of events is said to be pairwise independent.