STAT743 FOUNDATIONS OF STATISTICS (Part I)

Assignment 3 — Due at 2:30pm on November 14, 2019

Instructions:

1. Start each question on a new page and clearly indicate where each part of the question begins.

2. Submit questions in the same order as given below.

3. You are expected to show all details of your solution and any results taken from my notes or the textbook must be clearly and properly referenced.

4. No extensions to the due date and time will be given except in extreme circumstances and late assignments will not be accepted.

5. Students are reminded that submitted assignments must be their own work. Submission of someone else's solution (including solutions from the internet or other sources) under your name is academic misconduct and will be dealt with as such. Penalties for academic misconduct can include a 0 for the assignment, an F for the course with an annotation on your transcript and/or dismissal from your program of study.

Q. 1 **a)** Suppose that $X_1, \ldots, X_n$ is a random sample from a normal(μ, σ^2) distribution and let S^2 be the usual sample variance. Find the constant c such that $\hat{\sigma} = cS$ is an unbiased estimator of σ.

b) Suppose that $X_1, \ldots, X_n$ is a random sample from a population with mean μ and $\mathrm{E}(X^3) < \infty$. Let $\overline{X}$ and S^2 be the sample mean and variance. Show that

$$\mathrm{Cov}(\overline{X}, S^2) \;=\; \frac{1}{n}\,\mathrm{E}\left((X-\mu)^3\right)$$

Hint: It is easier to consider the random variable $Y = X - \mu$.

Q. 2 **a)** Derive the F probability density function by considering the ratio of two independent chi-squared random variables each divided by their respective degrees of freedom.

b) Derive the mean and variance of an $F_{p,q}$ random variable.
Hint: Express these moments in terms of expectations of powers of χ^2 *random variables*

c) Show the relationship between the F and Beta distributions

$$X \sim F_{p,q} \ \Rightarrow \ \frac{(p/q)X}{1+(p/q)X} \sim \text{Beta}(p/2, q/2)$$

Q. 3 **a)** Casella and Berger 5.19(a)

b) If Z_1 and Z_2 are independent standard normal random variables, show that the distribution of the minimum is not standard normal but the square of the minimum does have a χ^2_1 distribution.

c) Is it true that the square of a minimum of $n > 2$ independent standard normal random variables has a χ^2_1 distribution? Explain your answer.

Q. 4 **a)** Suppose that $X_1, \ldots, X_n$ is a random sample from a continuous distribution with pdf f and cdf F. Derive the joint pdf and cdf of $X_{(1)}$ and $X_{(n)}$ **without using Theorem 6.18 in my notes or 5.4.6 in the textbook**.

b) If $X_1, \ldots, X_n \overset{iid}{\sim}$ exponential(θ) show that the range $X_{(n)} - X_{(1)}$ is independent of $X_{(1)}$ and find the marginal distribution of each.

c) Casella and Berger 5.24. What is the distribution of $X_{(1)}/X_{(n)}$ in this case?