STAT743 FOUNDATIONS OF STATISTICS (Part I)

Assignment 4 Due at 12:30pm on Tuesday December 3, 2019

Instructions:

1. Start each question on a new page and clearly indicate where each part of the question begins.
2. Submit questions in the same order as given below.
3. You are expected to show all details of your solution and any results taken from my notes or the textbook must be clearly and properly referenced.
4. No extensions to the due date and time will be given except in extreme circumstances and late assignments will not be accepted.
5. Students are reminded that submitted assignments must be their own work. Submission of someone else's solution (including solutions from the internet or other sources) under your name is academic misconduct and will be dealt with as such. Penalties for academic misconduct can include a 0 for the assignment, an F for the course with an annotation on your transcript and/or dismissal from your program of study.

Q. 1 **a)** If $X_n \xrightarrow{p} 0$ and $Y_n \xrightarrow{p} 0$ then show that $X_n Y_n \xrightarrow{p} 0$.

b) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then show that $X_n + Y_n \xrightarrow{p} X + Y$.

c) Suppose that $Y_1, Y_2, \ldots$ is a sequence of random variables such that $\sqrt{n}(Y_n - \mu)$ converges in distribution to a normal$(0, \sigma^2)$ distribution, show that $Y_n \xrightarrow{p} \mu$.

Q. 2 **a)** Suppose that the sequence of random variables $\{X_n\}$ is bounded in probability and that $\{Y_n\}$ is another sequence of random variables such that $Y_n \xrightarrow{p} 0$ then prove that $X_n Y_n \xrightarrow{p} 0$.

b) Prove the second-order delta method which says that if $\sqrt{n}(Y_n - \theta) \xrightarrow{d} Z \sim \text{normal}(0, \sigma^2)$ and g is a function with $g'(\theta) = 0$ and $g''(\theta) \neq 0$ exists then

$$n\big(g(Y_n) - g(\theta)\big) \xrightarrow{d} \sigma^2 \frac{g''(\theta)}{2} X$$

where $X \sim \chi^2_1$.

Q. 3 **a)** Suppose that $X_1, X_2, \ldots$ are an infinite sequence of independent exponential random variables with mean parameter μ. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$

(i) Show that the sequence $X_{(n)}$ is not bounded in probability.

(ii) Now define another sequence of random variables $Z_n = X_{(n)} - \mu \log(n)$. Show that there exists a random variable Z such that $Z_n \xrightarrow{d} Z$. Give the cumulative distribution function and probability density function of Z.

The limiting distribution is called the Gumbel distribution and is important in modelling extreme values.

b) Casella and Berger 5.44

Q. 4 For the following question you may assume that we have a method to generate Uniform$(0, 1)$ random variables only.

a) Give an algorithm to generate random variables having a binomial distribution with parameters $n = 4$ and $p = 1/3$.

b) The probability density function of the logistic distribution is given on Page 624 of your textbook.

(i) Find the cumulative distribution function of the standard logistic distribution with $\mu = 0$ and $\beta = 1$ and use it to give an algorithm to generate observations from this distribution.

(ii) Generalize your algorithm to generate from a logistic distribution with arbitrary parameters $\mu \in \mathbb{R}$ and $\beta > 0$.

c) Casella and Berger 5.50