STAT743 FOUNDATIONS OF STATISTICS

Fall 2019

Assignment 1 — Solutions

Q. 1 **a)** $A = (A \bigcap B) \bigcup (A \bigcap B^c)$ and the sets on the right are mutually exclusive. Therefore by the third axiom we have $P(A) = P(A \bigcap B) + P(A \bigcap B^c)$. From the first axiom, we have that $P(A \bigcap B^c) \geqslant 0$ and so $P(A) \geqslant P(A \bigcap B)$. **[2 marks]**

Similarly we can write $A \bigcup B = A \bigcup (B \bigcap A^c)$ where the two sets on the right are mutually exclusive and so $P(A \bigcup B) = P(A) + P(B \bigcap A^c)$. The first axiom gives us $P(B \bigcap A^c) \geqslant 0$ and so $P(A \bigcup B) \geqslant P(A)$. Also the first part of this question gives us $P(B \bigcap A^c) \leqslant P(B)$ and so we have $P(A \bigcup B) \leqslant P(A) + P(B)$. **[4 marks]**

b) First we note that, for any two events, A and B we have

$$A \bigcup B \;=\; A \bigcup (A^c \bigcap B)$$

and that these two events on the right are mutually exclusive. so we have

$$\mathrm{P}(A \bigcup B) \;=\; \mathrm{P}(A) + \mathrm{P}(A^c \bigcap B)$$

Furthermore we have that

$$B \;=\; (A \bigcap B) \bigcup (A^c \bigcap B)$$

and the two events on the right are mutually exclusive so

$$\mathrm{P}(B) \;=\; \mathrm{P}(A \bigcap B) + \mathrm{P}(A^c \bigcap B) \quad \Rightarrow \quad \mathrm{P}(A^c \bigcap B) \;=\; \mathrm{P}(B) - \mathrm{P}(A \bigcap B)$$

and so, for any two events A and B we have

$$\mathrm{P}(A \bigcup B) \;=\; \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \bigcap B)$$

[4 marks]

We can extend this to three events as follows

$$\begin{aligned}
P(A\bigcup B\bigcup C) &= P((A\bigcup B)\bigcup C)\\
&= P(A\bigcup B) + P(C) - P((A\bigcup B)\bigcap C)\\
&= P(A) + P(B) - P(A\bigcap B) + P(C) - P((A\bigcap C)\bigcup(B\bigcap C))\\
&= P(A) + P(B) + P(C) - P(A\bigcap B)\\
&\quad -\Big[P(A\bigcap C) + P(B\bigcap C) - P((A\bigcap C)\bigcap(B\bigcap C))\Big].
\end{aligned}$$

Finally we note that $(A\bigcap C)\bigcap(B\bigcap C) = A\bigcap B\bigcap C$ and hence

$$P(A\bigcup B\bigcup C) = P(A)+P(B)+P(C)-P(A\bigcap B)-P(A\bigcap C)-P(B\bigcap C)+P(A\bigcap B\bigcap C)$$

[5 marks]

c) Casella and Berger 1.24

(i) Let E_i be the event that the game terminates (with a head) on the i^{th} toss. Clearly the E_i is a sequence of mutually exclusive events and we have

$$P(E_i) = P((i-1)\text{ tails followed by 1 head}) = \left(\frac{1}{2}\right)^i$$

A wins the game if the first head lands on a odd-numbered toss so

$$\begin{aligned}
P(\text{A wins}) &= P(E_1\bigcup E_3\bigcup E_5\bigcup\cdots)\\
&= P(E_1) + P(E_3) + P(E_5) + \cdots\\
&= \sum_{i=0}^{\infty} P(E_{2i+1})\\
&= \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{2i+1}\\
&= \sum_{i=0}^{\infty}\frac{1}{2}\left(\frac{1}{4}\right)^i\\
&= \frac{0.5}{1-0.25} = \frac{2}{3}
\end{aligned}$$

[5 marks]

The final result comes from the result for geometric series that

$$S = \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \quad \text{provided } |r| < 1$$

(ii) The only thing that changes when $p \neq 0.5$ is that

$$P(E_i) = (1-p)^{i-1}p$$

and so we get

$$P(\text{A wins}) = \sum_{i=0}^{\infty}(1-P)^{2i}p = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}$$

[3 marks]

(iii) Since $P(\text{A wins}) = \dfrac{1}{2-p}$ which is an increasing function of p for $p \in [0,1]$ we have

$$P(\text{A wins}) \geqslant \lim_{p \downarrow 0} \frac{1}{2-p} = \frac{1}{2}$$

[2 marks]

Q. 2 **a)** First we must show that

$$\frac{d^n}{dt^n} K_X(t) = \kappa_n(X) + \sum_{r=1}^{\infty} \frac{t^r}{r!} \kappa_{r+n}(X)$$

We will do this by induction. Consider $n = 1$

$$\begin{aligned} \frac{d}{dt} K_X(t) &= \sum_{r=1}^{\infty} \frac{d}{dt}\left(\frac{t^r}{r!}\right) \kappa_r(X) \\ &= \sum_{r=1}^{\infty} \frac{t^{r-1}}{(r-1)!} \kappa_r(X) \\ &= \kappa_1(X) + \sum_{r=1}^{\infty} \frac{t^r}{r!} \kappa_{r+1}(X) \end{aligned}$$

[3 marks]

Now suppose that

$$\frac{d^{n-1}}{dt^{n-1}} K_X(t) = \kappa_{n-1}(X) + \sum_{r=1}^{\infty} \frac{t^r}{r!} \kappa_{r+n-1}(X)$$

Then

$$\begin{aligned} \frac{d^n}{dt^n} K_X(t) &= \sum_{r=1}^{\infty} \frac{d}{dt}\left(\frac{t^r}{r!}\right) \kappa_{r+n-1}(X) \\ &= \sum_{r=1}^{\infty} \frac{t^{r-1}}{(r-1)!} \kappa_{r+n-1}(X) \\ &= \kappa_n(X) + \sum_{r=1}^{\infty} \frac{t^r}{r!} \kappa_{r+n}(X) \end{aligned}$$

[3 marks]

Hence our assertion is true and so setting $t = 0$ we see that

$$\left.\frac{d^r}{dt^r} K_X(t)\right|_{t=0} = \kappa_r(t) + \sum_{r=1}^{\infty} \frac{0^r}{r!} \kappa_{r+n}(X) = \kappa_r(t)$$

[1 mark]

b) It is easiest to use the original definition of $K_X(t)$ and the result of part a). To ease notation I will use $g'(t)$, $g''(t)$ and $g'''(t)$ to denote the first three derivatives of any function $g(t)$ with respect to t

The first three derivatives of $K_X(t)$ are

$$
\begin{aligned}
K'_X(t) &= \frac{d}{dt}\log\big(M_X(t)\big) \\
&= \frac{M'_X(t)}{M_X(t)} \\
K''_X(t) &= \frac{M''_X(t)}{M_X(t)} - \frac{\big(M'_X(t)\big)^2}{\left(M_X(t)\right)^2} \\
&= \frac{M''_X(t)}{M_X(t)} - \left(K'_X(t)\right)^2 \\
K'''_X(t) &= \frac{M'''_X(t)}{M_X(t)} - \frac{M'_X(t)M''_X(t)}{\big(M_X(t)\big)^2} - 2K'_X(t)K''_X(t)
\end{aligned}
$$

[4 marks]

Now recall that $M_X(0) = 1$ and that derivatives of $M_X(t)$ evaluated at $t = 0$ give the moments of X we have

$$
\begin{aligned}
K'_X(0) &= \frac{M'_X(0)}{M_X(0)} = \mu \\
K''_X(0) &= \frac{M''_X(0)}{M_X(0)} - \left(K'_X(0)\right)^2 \\
&= \mathrm{E}(X^2) - (\mu)^2 \\
&= \mathrm{E}\left(X^2 - 2\mu X + \mu^2\right) \\
&= \mathrm{E}\left((X-\mu)^2\right) \\
&= \mu_2 \\
K'''_X(0) &= \frac{M'''_X(0)}{M_X(0)} - \frac{M'_X(0)M''_X(0)}{\left(M_X(0)\right)^2} - 2K'_X(0)K''_X(0) \\
&= \mathrm{E}(X^3) - \mathrm{E}(X)\,\mathrm{E}(X^2) - 2\mu\mu_2 \\
&= \mathrm{E}(X^3) - \mathrm{E}(X)\,\mathrm{E}(X^2) - 2\mu\left(\mathrm{E}(X^2) - \mu^2\right) \\
&= \mathrm{E}(X^3) - 3\mu\,\mathrm{E}(X^2) + 2\mu^3 \\
&= \mathrm{E}\left(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right) \\
&= \mathrm{E}\left((X-\mu)^3\right) \\
&= \mu_3
\end{aligned}
$$

[6 marks]

c) If $X \sim \text{normal}(\mu, \sigma^2)$ then we know from the textbook (Page 625) that

$$M_X = \exp\left\{\mu t + \frac{1}{2}t^2\sigma^2\right\}$$

and so the cumulant generating function is

$$K_X(t) = \mu t + \frac{1}{2}t^2\sigma^2$$

[1 mark]

From this we see that the derivatives of the cumulant generating function are

$$\frac{d^r}{dt^r}K_X(t) = \begin{cases} \mu + t\sigma^2 & r = 1 \\ \sigma^2 & r = 2 \\ 0 & r = 3, 4, \ldots \end{cases}$$

[2 marks]

Hence we see that

$$\kappa_r(X) = \begin{cases} \mu & r = 1 \\ \sigma^2 & r = 2 \\ 0 & r = 3, 4, \ldots \end{cases}$$

[1 mark]

d) By Theorem 4.5 in my notes (Theorem 4.2.12 in the textbook) we know that

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

because X and Y are independent random variables.

Hence we have

$$\begin{aligned} \kappa_r(X+Y) &= \left.\frac{d^r}{dt^r}K_{X+Y}(t)\right|_{t=0} \\ &= \left.\frac{d^r}{dt^r}\log\{M_{X+Y}(t)\}\right|_{t=0} \\ &= \left.\frac{d^r}{dt^r}\log\{M_X(t)M_Y(t)\}\right|_{t=0} \\ &= \left.\frac{d^r}{dt^r}\{K_X(t)+K_Y(t)\}\right|_{t=0} \\ &= \left.\frac{d^r}{dt^r}K_X(t)\right|_{t=0} + \left.\frac{d^r}{dt^r}K_Y(t)\right|_{t=0} \\ &= \kappa_r(X) + \kappa_r(Y) \end{aligned}$$

[4 marks]

Q. 3 a)

$$\begin{aligned} M_Y(t) &= \mathrm{E}\left(\mathrm{e}^{tY}\right) \\ &= \sum_{y=0}^{\infty} \mathrm{e}^{ty} \binom{y+r-1}{r-1} p^r (1-p)^y \\ &= \sum_{y=0}^{\infty} \binom{y+r-1}{r-1} p^r \left((1-p)\mathrm{e}^t\right)^y \\ &= \left(\frac{p}{1-(1-p)\mathrm{e}^t}\right)^r \sum_{y=0}^{\infty} \binom{y+r-1}{r-1} \left(1-(1-p)\mathrm{e}^t\right)^r \left((1-p)\mathrm{e}^t\right)^y \end{aligned}$$

The summand in the above expression is the negative binomial pmf with parameters r and $1-(1-p)\mathrm{e}^t$ provided that $0 < 1-(1-p)\mathrm{e}^t < 1$. and so for, provided this is true, the infinite sum equals 1. **[6 marks]**

Now

$$0 < 1-(1-p)\mathrm{e}^t < 1 \iff 0 < (1-p)\mathrm{e}^t < 1 \iff \log(1-p)+t < 0 \iff t < -\log(1-p)$$

and we note that, since $1-p < 1$, $-\log(1-p) > 0$.

Hence,

$$M_Y(t) = \left(\frac{p}{1-(1-p)\mathrm{e}^t}\right)^r \qquad \text{for } t \ < \ -\log(1-p)$$

[2 marks]

b)

$$\begin{aligned} M_Y(t) &= p^r \left(1-(1-p)\mathrm{e}^t\right)^{-r} \\ \Rightarrow M_Y'(t) &= rp^r \left(1-(1-p)\mathrm{e}^t\right)^{-r-1} (1-p)\mathrm{e}^t \\ \Rightarrow M_Y''(t) &= r(r+1)p^r \left(1-(1-p)\mathrm{e}^t\right)^{-r-2} (1-p)^2\mathrm{e}^{2t} \\ &\quad + rp^r \left(1-(1-p)\mathrm{e}^t\right)^{-r-1} (1-p)\mathrm{e}^t \end{aligned}$$

[2 marks]

Hence we can get the moments

$$E(Y) = M_Y'(0) \ = \ rp^r(1-(1-p))^{-r-1}(1-p) = \frac{r(1-p)}{p}$$

[2 marks]

$$
\begin{aligned}
E(Y^2) = M_Y''(0) &= r(r+1)p^r(1-(1-p))^{-r-2}(1-p)^2 + \frac{r(1-p)}{p} \\
&= \frac{r(r+1)(1-p)^2}{p^2} + \frac{r(1-p)}{p} \\
&= \frac{r(1-p)}{p^2}\big((r+1)(1-p)+p\big) \\
&= \frac{r(1-p)(1+r-pr)}{p^2} \\
\Rightarrow \quad \mathrm{Var}(Y) &= \frac{r(1-p)(1+r-pr)}{p^2} - \left(\frac{r(1-p)}{p}\right)^2 \\
&= \frac{r(1-p)}{p^2}\big(1+r-pr-r(1-p)\big) \\
&= \frac{r(1-p)}{p^2}
\end{aligned}
$$

[4 marks]

c) To show that the distribution of $Z = pY$ converges as $p \to 0$ all we need to verify is convergence of moment generating functions because of Theorem 2.11 in my notes (Theorem 2.3.12 in the textbook).

The moment generating function of Z is

$$
\begin{aligned}
M_Z(t) &= \mathrm{E}\left(e^{tZ}\right) \\
&= \mathrm{E}\left(e^{tpY}\right) \\
&= M_Y(tp) \\
&= \left(\frac{p}{1-(1-p)e^{pt}}\right)^r
\end{aligned}
$$

[3 marks]

Hence taking limits as $p \to 0$ we have

$$
\begin{aligned}
\lim_{p\to 0} M_Z(t) &= \lim_{p\to 0}\left(\frac{p}{1-(1-p)e^{pt}}\right)^r \\
&= \left(\lim_{p\to 0}\frac{p}{1-(1-p)e^{pt}}\right)^r \\
&= \left(\lim_{p\to 0}\frac{1}{e^{pt}-t(1-p)e^{pt}}\right)^r \\
&= (1-t)^{-r}
\end{aligned}
$$

[3 marks]

Now if $X \sim \text{gamma}(\alpha, \beta)$ then

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-(1/\beta - t)x}\, dx \\ &= \left(\frac{1}{\beta(1/\beta - t)}\right)^\alpha \int_0^\infty \frac{(1/\beta - t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(1/\beta - t)x}\, dx \end{aligned}$$

Now for $t < 1/\beta$, the integrand is the pdf of a gamma random variable with parameters α and $(1/\beta - t)^{-1}$ and so integrates to 1. Thus the moment generating function of a gamma random variable is

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

Comparing this to the limit of moment generating functions found above we see that the limit is a moment generating function of a gamma random variable with parameters $\alpha = r$ and $\beta = 1$. Since the moment generating functions converge, so do the cumulative distribution functions and so we can say that Z *converges in distribution* to a $\text{gamma}(\alpha = r, \beta = 1)$ random variable as $p \to 0$. **[3 marks]**

Q. 4 **a)** Suppose that Y has a log-normal distribution then from the definition we see that we can write

$$Y = e^X \quad \text{where} \quad X \sim \text{Normal}(\mu, \sigma^2)$$

The easy way to get the moments of Y is from the moment generating function of X because

$$E\left(Y^r\right) = E\left(e^{rX}\right) = M_X(r) = \exp\left\{r\mu + \frac{1}{2}r^2\sigma^2\right\}$$

[2 marks]

Hence we have

$$\begin{aligned}
E(Y) &= \exp\left\{\mu + \frac{\sigma^2}{2}\right\} \\
E(Y^2) &= \exp\left\{2\mu + 2\sigma^2\right\} \\
\text{Var}(Y) &= \exp\left\{2\mu + 2\sigma^2\right\} - \exp\left\{2\left(\mu + \frac{\sigma^2}{2}\right)\right\} = e^{2\mu}\left(e^{2\sigma^2} - e^{\sigma^2}\right)
\end{aligned}$$

[3 marks]

b) For convenience, I will assume that X is a continuous random variable although that is not necessary and the proof is identical in the discrete case.

First we note that, as for all pdfs,

$$\int_{-\infty}^{\infty} h(x)c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx = 1$$

Since this is a constant for all $\boldsymbol{\eta}$ its partial derivatives must be 0 and so we have

$$\frac{\partial}{\partial \eta_j}\int_{-\infty}^{\infty} h(x)c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx = 0$$

For the exponential family we can always interchange integration and differentiation and so let us do this (also using the chain rule inside the integrand) to get

$$\int_{-\infty}^{\infty} h(x)\left(\frac{\partial}{\partial \eta_j}c^*(\boldsymbol{\eta})\right) \exp\left\{\sum \eta_i t_i(x)\right\} dx + \int_{-\infty}^{\infty} t_j(x)h(x)c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx = 0$$

[3 marks]

The second term in the sum on the left of the above expression is $E\left(t_j(X)\right)$ by definition of expectation so we have

$$E\left(t_j(X)\right) = -\int_{-\infty}^{\infty} h(x)\left(\frac{\partial}{\partial \eta_j}c^*(\boldsymbol{\eta})\right) \exp\left\{\sum \eta_i t_i(x)\right\} dx$$

[1 mark]

Now recall that

$$\frac{\partial}{\partial \eta_j} \log c^*(\boldsymbol{\eta}) = \frac{\frac{\partial}{\partial \eta_j} c^*(\boldsymbol{\eta})}{c^*(\boldsymbol{\eta})} \;\Rightarrow\; \frac{\partial}{\partial \eta_j} c^*(\boldsymbol{\eta}) = c^*(\boldsymbol{\eta}) \frac{\partial}{\partial \eta_j} \log c^*(\boldsymbol{\eta})$$

Applying this in the above integrand we have

$$\begin{aligned}
\mathrm{E}\left(t_j(X)\right) &= -\int_{-\infty}^{\infty} h(x) \left(\frac{\partial}{\partial \eta_j} \log c^*(\boldsymbol{\eta})\right) c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx \\
&= -\left(\frac{\partial}{\partial \eta_j} \log c^*(\boldsymbol{\eta})\right) \int_{-\infty}^{\infty} h(x) c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx \\
&= -\frac{\partial}{\partial \eta_j} \log c^*(\boldsymbol{\eta})
\end{aligned}$$

since the integrand is now that of the original pdf. **[4 marks]**

To prove the second part we only need to show that

$$\frac{\partial}{\partial \eta_j} \mathrm{E}\left(t_j(X)\right) = \mathrm{Var}\left(t_j(X)\right)$$

$$\begin{aligned}
\frac{\partial}{\partial \eta_j} \mathrm{E}\left(t_j(X)\right) &= \frac{\partial}{\partial \eta_j} \int_{-\infty}^{\infty} t_j(x) h(x) c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx \\
&= \int_{-\infty}^{\infty} t_j(x) h(x) \left(\frac{\partial}{\partial \eta_j} c^*(\boldsymbol{\eta})\right) \exp\left\{\sum \eta_i t_i(x)\right\} dx \\
&\quad + \int_{-\infty}^{\infty} t_j^2(x) h(x) c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx \\
&= \left(\frac{\partial}{\partial \eta_j} \log c^*(\boldsymbol{\eta})\right) \int_{-\infty}^{\infty} t_j(x) h(x) c^*(\boldsymbol{\eta}) \exp\left\{\sum \eta_i t_i(x)\right\} dx + E[t_j^2(X)] \\
&= \left[-\mathrm{E}\left(t_j(X)\right)\right] \mathrm{E}\left(t_j(X)\right) + E\left(t_j^2(X)\right)
\end{aligned}$$

Where the first expectation is a consequence of the result proved above and the other two are from the definition of expectation. **[3 marks]**

Hence we have

$$\frac{\partial}{\partial \eta_j} \mathrm{E}\left(t_j(X)\right) = -\left[\mathrm{E}\left(t_j(X)\right)\right]^2 + E\left(t_j^2(X)\right) = \mathrm{Var}(t_j(X))$$

[1 mark]

The other part of the result is simply that following from our first result

$$\frac{\partial}{\partial \eta_j} \mathrm{E}\left(t_j(X)\right) = \frac{\partial}{\partial \eta_j}\left(-\frac{\partial}{\partial \eta_j} \log c^*(\boldsymbol{\eta})\right) = -\frac{\partial^2}{\partial \eta_j^2} \log c^*(\boldsymbol{\eta})$$

[1 mark]

c) I will do the Poisson first as it is a little easier. In that case we have the probability mass function

$$f(x \mid \lambda) \;=\; \frac{\lambda^x e^{-\lambda} I(x \in \{0,1,2,\ldots\})}{x!} \;=\; \frac{I(x \in \{0,1,2,\ldots\})}{x!} e^{-\lambda} e^{x \log \lambda}$$

Hence we can take the natural parameter $\eta \;=\; \log \lambda$ and corresponding $t_1(x) = x$.

[2 marks]

Clearly we have

$$h(x) \;=\; \frac{I(x \in \{0,1,2,\ldots\})}{x!}$$

so all that is left is to find $c^*(\eta)$. We note that $\eta = \log \lambda$ so we have $\lambda = e^\eta$ and since $c(\lambda) = e^{-\lambda}$ we get

$$c^*(\eta) \;=\; \exp\{-e^\eta\}$$

[2 marks]

And so we can write $f(x \mid \eta) \;=\; h(x)c^*(\eta)\exp\{t(x)\eta\}$ as required

For the normal we can proceed in a similar way except that there is now a vector parameter $\theta = (\mu, \sigma^2)$. We can write the density function as

$$\begin{aligned} f(x \mid \mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right\} \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{e^{-\mu^2/2\sigma^2}}{\sigma}\right) \exp\left\{x\frac{\mu}{\sigma^2} - \frac{x^2}{2}\left(\frac{1}{\sigma^2}\right)\right\} \end{aligned}$$

Hence we get the natural parameters

$$\eta_1 \;=\; \frac{\mu}{\sigma^2} \qquad \eta_2 \;=\; \frac{1}{\sigma^2}$$

with corresponding $t_1(x) = x$, $t_2(x) = -x^2/2$. **[2 marks]**

Outside of the exponent we have $h(x) = 1/\sqrt{2\pi}$ and

$$c(\theta) \;=\; \frac{e^{-\mu^2/2\sigma^2}}{\sigma}$$

We can express this in terms of the natural parameters by

$$c^*(\eta_1, \eta_2) \;=\; \sqrt{\eta_2 e^{-\eta_1^2/\eta_2}}$$

[2 marks]

Hence we can write the normal pdf in the canonical exponential family form

$$f(x \mid \eta_1, \eta_2) \;=\; h(x)c^*(\eta_1, \eta_2)\exp\{t_1(x)\eta_1 + t_2(x)\eta_2\}$$